CHAPTER VIII.

OF THE MOVEMENT OF HEAT IN A SOLID CUBE.

333. It still remains for us to make use of the equation
\[ \frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} \right) \]
which represents the movement of heat in a solid cube exposed to the action of the air (Chapter II., Section v.). Assuming, in the first place, for \( v \) the very simple value \( e^{-mt} \cos nx \cos py \cos qz \), if we substitute it in the proposed equation, we have the equation of condition \( m = k(n^2 + p^2 + q^2) \), the letter \( k \) denoting the coefficient \( \frac{K}{CD} \). It follows from this that if we substitute for \( n, p, q \) any quantities whatever, and take for \( m \) the quantity \( k(n^2 + p^2 + q^2) \), the preceding value of \( v \) will always satisfy the partial differential equation. We have therefore the equation
\[ v = e^{-k(n^2+p^2+q^2)t} \cos nx \cos py \cos qz. \]
The nature of the problem requires also that if \( x \) changes sign, and if \( y \) and \( z \) remain the same, the function should not change; and that this should also hold with respect to \( y \) or \( z \): now the value of \( v \) evidently satisfies these conditions.

334. To express the state of the surface, we must employ the following equations:
\[ \pm K \frac{dv}{dx} + hv = 0 \]
\[ \pm K \frac{dv}{dy} + hv = 0 \]
\[ \pm K \frac{dv}{dz} + hv = 0 \]

\[ \text{....................}(b). \]
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These ought to be satisfied when \( x = \pm a \), or \( y = \pm a \), or \( z = \pm a \). The centre of the cube is taken to be the origin of co-ordinates; and the side is denoted by \( a \).

The first of the equations (b) gives

\[ \mp e^{-mt} n \sin nx \cos py \cos qz + \frac{h}{K} \cos nx \cos py \cos qz = 0, \]

or

\[ \mp n \tan nx + \frac{h}{K} = 0, \]

an equation which must hold when \( x = \pm a \).

It follows from this that we cannot take any value whatever for \( n \), but that this quantity must satisfy the condition

\[ na \tan na = \frac{h}{K} a. \]

We must therefore solve the definite equation

\[ e \tan e = \frac{h}{K} a, \]

which gives the value of \( e \), and take \( n = \frac{e}{a} \). Now the equation in \( e \) has an infinity of real roots; hence we can find for \( n \) an infinity of different values. We can ascertain in the same manner the values which may be given to \( p \) and to \( q \); they are all represented by the construction which was employed in the preceding problem (Art. 321). Denoting these roots by \( n_1, n_2, n_3, \&c. \); we can then give to \( v \) the particular value expressed by the equation

\[ v = e^{-kt(n^2+p^2+q^2)} \cos nx \cos py \cos qz, \]

provided we substitute for \( n \) one of the roots \( n_1, n_2, n_3, \&c. \), and select \( p \) and \( q \) in the same manner.

325. We can thus form an infinity of particular values of \( v \), and it evident that the sum of several of these values will also satisfy the differential equation (a), and the definite equations (b). In order to give to \( v \) the general form which the problem requires, we may unite an indefinite number of terms similar to the term

\[ ae^{-kt(n^2+p^2+q^2)} \cos nx \cos py \cos qz. \]

The value of \( v \) may be expressed by the following equation:

\[ v = (a_1 \cos n_1 x e^{-kn_1 t} + a_2 \cos n_2 x e^{-kn_2 t} + a_3 \cos n_3 x e^{-kn_3 t} + \&c.), \]

\[ (b_1 \cos n_1 y e^{-kn_1 t} + b_2 \cos n_2 y e^{-kn_2 t} + b_3 \cos n_3 y e^{-kn_3 t} + \&c.), \]

\[ (c_1 \cos n_1 z e^{-kn_1 t} + c_2 \cos n_2 z e^{-kn_2 t} + c_3 \cos n_3 y e^{-kn_3 t} + \&c.). \]
The second member is formed of the product of the three factors written in the three horizontal lines, and the quantities $a_1$, $a_2$, $a_3$, &c. are unknown coefficients. Now, according to the hypothesis, if $t$ be made $= 0$, the temperature must be the same at all points of the cube. We must therefore determine $a_1$, $a_2$, $a_3$, &c., so that the value of $v$ may be constant, whatever be the values of $x$, $y$, and $z$, provided that each of these values is included between $a$ and $-a$. Denoting by $1$ the initial temperature at all points of the solid, we shall write down the equations (Art. 323)

$$1 = a_1 \cos n_x x + a_2 \cos n_y y + a_3 \cos n_z z + \&c.,$$
$$1 = b_1 \cos n_x y + b_2 \cos n_y y + b_3 \cos n_z y + \&c.,$$
$$1 = c_1 \cos n_x z + c_2 \cos n_y z + c_3 \cos n_z z + \&c.,$$

in which it is required to determine $a_1$, $a_2$, $a_3$, &c. After multiplying each member of the first equation by $\cos nx$, integrate from $x=0$ to $x=a$: it follows then from the analysis formerly employed (Art. 324) that we have the equation

$$1 = \frac{\sin n_x a \cos n_x}{\frac{1}{2} n_x (1 + \sin \frac{2 n_x}{2 n_x})} + \frac{\sin n_y a \cos n_y}{\frac{1}{2} n_y (1 + \sin \frac{2 n_y}{2 n_y})} + \frac{\sin n_z a \cos n_z}{\frac{1}{2} n_z (1 + \sin \frac{2 n_z}{2 n_z})} + \&c.$$

Denoting by $\mu$, the quantity $\frac{1}{2} \left(1 + \frac{\sin 2 n_x}{2 n_x}\right)$, we have

$$1 = \frac{\sin n_x a \cos n_x}{n_x \mu} + \frac{\sin n_y a \cos n_x}{n_y a \mu_2} + \frac{\sin n_z a \cos n_x}{n_z a \mu_3} + \&c.$$  

This equation holds always when we give to $x$ a value included between $a$ and $-a$.

From it we conclude the general value of $v$, which is given by the following equation

$$v = \left(\frac{\sin n_x a \cos n_x e^{-n_x t}}{n_x \mu_1} + \frac{\sin n_y a \cos n_y e^{-n_y t}}{n_y a \mu_2} + \&c.,\right)$$

$$\left(\frac{\sin n_x a \cos n_y e^{-n_y t}}{n_y a \mu_1} + \frac{\sin n_y a \cos n_y e^{-n_y t}}{n_y a \mu_2} + \&c.,\right)$$

$$\left(\frac{\sin n_x a \cos n_z e^{-n_z t}}{n_z a \mu_1} + \frac{\sin n_y a \cos n_y e^{-n_y t}}{n_y a \mu_2} + \&c.,\right).$$
336. The expression for \( v \) is therefore formed of three similar functions, one of \( x \), the other of \( y \), and the third of \( z \), which is easily verified directly.

In fact, if in the equation

\[
\frac{dv}{dt} = k \left( \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} \right),
\]

we suppose \( v = XYZ \); denoting by \( X \) a function of \( x \) and \( t \), by \( Y \) a function of \( y \) and \( t \), and by \( Z \) a function of \( z \) and \( t \), we have

\[
YZ = \frac{dX}{dt} + ZX \frac{dY}{dt} + XY \frac{dZ}{dt} = k \left( YZ \frac{d^2X}{dx^2} + ZX \frac{d^2Y}{dy^2} + XY \frac{d^2Z}{dz^2} \right),
\]

or

\[
\frac{1}{X} \frac{dX}{dt} + \frac{1}{Y} \frac{dY}{dt} + \frac{1}{Z} \frac{dZ}{dt} = k \left( \frac{1}{X} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} + \frac{1}{Z} \frac{d^2Z}{dz^2} \right),
\]

which implies the three separate equations

\[
\frac{dX}{dt} = k \frac{d^2X}{dx^2}, \quad \frac{dY}{dt} = k \frac{d^2Y}{dy^2}, \quad \frac{dZ}{dt} = k \frac{d^2Z}{dz^2}.
\]

We must also have as conditions relative to the surface,

\[
\frac{dV}{dx} + \frac{k}{K} V = 0, \quad \frac{dV}{dy} + \frac{k}{K} V = 0, \quad \frac{dV}{dz} + \frac{k}{K} V = 0,
\]

whence we deduce

\[
\frac{dX}{dx} + \frac{h}{K} X = 0, \quad \frac{dY}{dy} + \frac{h}{K} Y = 0, \quad \frac{dZ}{dz} + \frac{h}{K} Z = 0.
\]

It follows from this, that, to solve the problem completely, it is enough to take the equation \( \frac{du}{dt} = k \frac{d^2u}{dx^2} \), and to add to it the equation of condition \( \frac{du}{dx} + \frac{h}{K} u = 0 \), which must hold when \( x = a \).

We must then put in the place of \( x \), either \( y \) or \( z \), and we shall have the three functions \( X, Y, Z \), whose product is the general value of \( v \).

Thus the problem proposed is solved as follows:

\[
v = \phi (x, t) \phi (y, t) \phi (z, t);
\]

\[
\phi (x, t) = \sin \frac{n_x a}{n_1 \mu_1} \cos n_x e^{-kn_1 t} + \sin \frac{n_x a}{n_2 \mu_2} \cos n_x e^{-kn_2 t} + \sin \frac{n_x a}{n_3 \mu_3} \cos n_x e^{-kn_3 t} + \&c.;
\]
n_1, n_2, n_3, &c. being given by the following equation

\[ \epsilon \tan \epsilon = \frac{ha}{K}, \]

in which \( \epsilon \) represents \( na \) and the value of \( \mu_i \) is

\[ \frac{1}{2} \left( 1 + \frac{\sin 2n_i \alpha}{2n_i \alpha} \right). \]

In the same manner the functions \( \phi (y, t), \phi (z, t) \) are found.

337. We may be assured that this value of \( v \) solves the problem in all its extent, and that the complete integral of the partial differential equation (a) must necessarily take this form in order to express the variable temperatures of the solid.

In fact, the expression for \( v \) satisfies the equation (a) and the conditions relative to the surface. Hence the variations of temperature which result in one instant from the action of the molecules and from the action of the air on the surface, are those which we should find by differentiating the value of \( v \) with respect to the time \( t \). It follows that if, at the beginning of any instant, the function \( v \) represents the system of temperatures, it will still represent those which hold at the commencement of the following instant, and it may be proved in the same manner that the variable state of the solid is always expressed by the function \( v \), in which the value of \( t \) continually increases. Now this function agrees with the initial state: hence it represents all the later states of the solid. Thus it is certain that any solution which gives for \( v \) a function different from the preceding must be wrong.

338. If we suppose the time \( t \), which has elapsed, to have become very great, we no longer have to consider any but the first term of the expression for \( v \); for the values \( n_1, n_2, n_3, \&c. \) are arranged in order beginning with the least. This term is given by the equation

\[ v = \left( \frac{\sin n_1 \alpha}{n_1 \alpha \mu_1} \right)^3 \cos n_1 x \cos n_1 y \cos n_1 z e^{-3in_1 \alpha t}; \]

this then is the principal state towards which the system of temperatures continually tends, and with which it coincides without sensible error after a certain value of \( t \). In this state the tempe-
rature at every point decreases proportionally to the powers of
the fraction \( e^{-5k_n^2} \); the successive states are then all similar, or
rather they differ only in the magnitudes of the temperatures
which all diminish as the terms of a geometrical progression, pre-
serving their ratios. We may easily find, by means of the pre-
ceding equation, the law by which the temperatures decrease from
one point to another in direction of the diagonals or the edges of
the cube, or lastly of a line given in position. We might ascer-
tain also what is the nature of the surfaces which determine the
layers of the same temperature. We see that in the final and
regular state which we are here considering, points of the same
layer preserve always equal temperatures, which would not hold
in the initial state and in those which immediately follow it.
During the infinite continuance of the ultimate state the mass is
divided into an infinity of layers all of whose points have a com-
mon temperature.

339. It is easy to determine for a given instant the mean
temperature of the mass, that is to say, that which is obtained by
taking the sum of the products of the volume of each molecule
by its temperature, and dividing this sum by the whole volume.
We thus form the expression \( \iiint \frac{vdx dy dz}{2^3 \alpha^3} \), which is that of the
mean temperature \( V \). The integral must be taken successively
with respect to \( x, y, \) and \( z \), between the limits \( a \) and \(-a\); \( v \) being
equal to the product \( X Y Z \), we have

\[
V = \int Xdx \int Ydy \int Zdz;
\]
thus the mean temperature is \( \left( \frac{\int Xdx}{2a} \right)^3 \), since the three complete
integrals have a common value, hence

\[
\sqrt[3]{V} = \left( \frac{\sin n_1 a}{n_1 a} \right)^2 \frac{1}{\mu_1} e^{-kn_1 \xi} + \left( \frac{\sin n_2 a}{n_2 a} \right)^2 \frac{1}{\mu_2} e^{-kn_2 \xi} + \&c.
\]

The quantity \( na \) is equal to \( \epsilon \), a root of the equation \( \epsilon \tan \epsilon = \frac{ha}{K} \),
and \( \mu \) is equal to \( \frac{1}{2} \left( 1 + \frac{\sin 2\epsilon}{2\epsilon} \right) \). We have then, denoting the
different roots of this equation by \( \epsilon_1, \epsilon_2, \epsilon_3, \&c. \),
\[ 2 \sqrt{V} = \left( \frac{\sin \epsilon_1}{\epsilon_1} \right)^2 \frac{e^{-k \epsilon_1^2 t}}{1 + \frac{\sin 2\epsilon_1}{2\epsilon_1}} + \left( \frac{\sin \epsilon_2}{\epsilon_2} \right)^2 \frac{e^{-k \epsilon_2^2 t}}{1 + \frac{\sin 2\epsilon_2}{2\epsilon_2}} + \&c. \]

\( \epsilon_1 \) is between 0 and \( \frac{\pi}{2} \), \( \epsilon_2 \) is between \( \pi \) and \( \frac{3\pi}{2} \), \( \epsilon_3 \) between \( 2\pi \) and \( \frac{5\pi}{2} \), the roots \( \epsilon_4, \epsilon_5, \epsilon_6, \&c. \) approach more and more nearly to the inferior limits \( \pi, 2\pi, 3\pi, \&c. \), and end by coinciding with them when the index \( i \) is very great. The double arcs \( 2\epsilon_1, 2\epsilon_2, 2\epsilon_3, \&c. \) are included between 0 and \( \pi, \) between \( 2\pi \) and \( 3\pi, \) between \( 4\pi \) and \( 5\pi; \) for which reason the sines of these arcs are all positive: the quantities \( 1 + \frac{\sin 2\epsilon_1}{2\epsilon_1}, 1 + \frac{\sin 2\epsilon_2}{2\epsilon_2}, \&c., \) are positive and included between 1 and 2. It follows from this that all the terms which enter into the value of \( \sqrt{V} \) are positive.

340. We propose now to compare the velocity of cooling in the cube, with that which we have found for a spherical mass. We have seen that for either of these bodies, the system of temperatures converges to a permanent state which is sensibly attained after a certain time; the temperatures at the different points of the cube then diminish all together preserving the same ratios, and the temperatures of one of these points decrease as the terms of a geometric progression whose ratio is not the same in the two bodies. It follows from the two solutions that the ratio for the sphere is \( e^{-kn^2} \) and for the cube \( e^{-3e^2a^2k}. \) The quantity \( n \) is given by the equation

\[ na \frac{\cos na}{\sin na} = 1 - \frac{h}{K} a, \]

\( a \) being the semi-diameter of the sphere, and the quantity \( e \) is given by the equation \( e \tan e = \frac{h}{K} a, \) \( a \) being the half side of the cube.

This arranged, let us consider two different cases; that in which the radius of the sphere and the half side of the cube are each equal to \( a, \) a very small quantity; and that in which the value of \( a \) is very great. Suppose then that the two bodies are of
small dimensions; \( \frac{ha}{K} \) having a very small value, the same is the case with \( e \), we have therefore \( \frac{ha}{K} = e^a \), hence the fraction

\[
e^{-\frac{a^2}{\alpha^2}} \text{ is equal to } e^{-\frac{3h}{CDa}}.
\]

Thus the ultimate temperatures which we observe are expressed in the form \( A e^{-\frac{3ht}{CDa}} \). If now in the equation \( \frac{na \cos na}{\sin na} = 1 - \frac{h}{K} a \), we suppose the second member to differ very little from unity, we find

\[
\frac{h}{K} = \frac{n^2a}{3},
\]

hence the fraction \( e^{-kn^2} \) is \( e^{-\frac{3h}{CDa}} \).

We conclude from this that if the radius of the sphere is very small, the final velocities of cooling are the same in that solid and in the circumscribed cube, and that each is in inverse ratio of the radius; that is to say, if the temperature of a cube whose half side is \( a \) passes from the value \( A \) to the value \( B \) in the time \( t \), a sphere whose semi-diameter is \( a \) will also pass from the temperature \( A \) to the temperature \( B \) in the same time. If the quantity \( a \) were changed for each body so as to become \( a' \), the time required for the passage from \( A \) to \( B \) would have another value \( t' \), and the ratio of the times \( t \) and \( t' \) would be that of the half sides \( a \) and \( a' \). The same would not be the case when the radius \( a \) is very great: for \( e \) is then equal to \( \frac{1}{2\pi} \), and the values of \( na \) are the quantities \( \pi, 2\pi, 3\pi, 4\pi, \&c. \)

We may then easily find, in this case, the values of the fractions \( e^{-\frac{a^2}{a^2}}, e^{-kn^2} \); they are \( e^{-\frac{3\pi^2}{4a^2}} \) and \( e^{-\frac{kn^2}{a^2}} \).

From this we may derive two remarkable consequences: 1st, when two cubes are of great dimensions, and \( a \) and \( a' \) are their half-sides; if the first occupies a time \( t \) in passing from the temperature \( A \) to the temperature \( B \), and the second the time \( t' \) for the same interval; the times \( t \) and \( t' \) will be proportional to the squares \( a^2 \) and \( a'^2 \) of the half-sides. We found a similar result for spheres of great dimensions. 2nd, If the length \( a \) of the half-side of a cube is considerable, and a sphere has the same magnitude \( a \) for radius, and during the time \( t \) the temperature of the cube falls from \( A \) to \( B \), a different time \( t' \) will elapse whilst the temperature of the
sphere is falling from $A$ to $B$, and the times $t$ and $t'$ are in the ratio of 4 to 3.

Thus the cube and the inscribed sphere cool equally quickly when their dimension is small; and in this case the duration of the cooling is for each body proportional to its thickness. If the dimension of the cube and the inscribed sphere is great, the final duration of the cooling is not the same for the two solids. This duration is greater for the cube than for the sphere, in the ratio of 4 to 3, and for each of the two bodies severally the duration of the cooling increases as the square of the diameter.

341. We have supposed the body to be cooling slowly in atmospheric air whose temperature is constant. We might submit the surface to any other condition, and imagine, for example, that all its points preserve, by virtue of some external cause, the fixed temperature 0. The quantities $n, p, q$, which enter into the value of $v$ under the symbol cosine, must in this case be such that $\cos nx$ becomes nothing when $x$ has its complete value $a$, and that the same is the case with $\cos py$ and $\cos qz$. If $2a$ the side of the cube is represented by $\pi$, $2\pi$ being the length of the circumference whose radius is 1; we can express a particular value of $v$ by the following equation, which satisfies at the same time the general equation of movement of heat, and the state of the surface,

$$v = e^{-\frac{Kt}{CD}} \cos x \cdot \cos y \cdot \cos z.$$

This function is nothing, whatever be the time $t$, when $x$ or $y$ or $z$ receive their extreme values $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$: but the expression for the temperature cannot have this simple form until after a considerable time has elapsed, unless the given initial state is itself represented by $\cos x \cos y \cos z$. This is what we have supposed in Art. 100, Sect. VIII. Chap. I. The foregoing analysis proves the truth of the equation employed in the Article we have just cited.

Up to this point we have discussed the fundamental problems in the theory of heat, and have considered the action of that element in the principal bodies. Problems of such kind and order have been chosen, that each presents a new difficulty of a higher degree. We have designedly omitted a numerous variety of
intermediate problems, such as the problem of the linear movement of heat in a prism whose ends are maintained at fixed temperatures, or exposed to the atmospheric air. The expression for the varied movement of heat in a cube or rectangular prism which is cooling in an aëriform medium might be generalised, and any initial state whatever supposed. These investigations require no other principles than those which have been explained in this work.

A memoir was published by M. Fourier in the Mémoires de l'Académie des Sciences, Tome vii. Paris, 1827, pp. 605—624, entitled, Mémoire sur la distinction des racines imaginaires, et sur l'application des théorèmes d'analyse algébrique aux équations transcendantes qui dépendent de la théorie de la chaleur. It contains a proof of two propositions in the theory of heat. If there be two solid bodies of similar convex forms, such that corresponding elements have the same density, specific capacity for heat, and conductivity, and the same initial distribution of temperature, the condition of the two bodies will always be the same after times which are as the squares of the dimensions, when, 1st, corresponding elements of the surfaces are maintained at constant temperatures, or 2nd, when the temperatures of the exterior medium at corresponding points of the surface remain constant.

For the velocities of flow along lines of flow across the terminal areas s, s' of corresponding prismatic elements are as \( u - v : u' - v' \), where \( (u, v), (u', v') \) are temperatures at pairs of points at the same distance \( \frac{1}{2} \Delta \) on opposite sides of \( s \) and \( s' \); and if \( n : n' \) is the ratio of the dimensions, \( u - v : u' - v' = n' : n \). If then, \( dt, dt' \) be corresponding times, the quantities of heat received by the prismatic elements are as \( sk(u - v)dt : s'k(u' - v')dt' \), or as \( n^n'dt : n'^n'dt' \). But the volumes being as \( n^3 : n'^3 \), if the corresponding changes of temperature are always equal we must have

\[
\frac{n^2'n'dt}{n^3} = \frac{n'^2vdt'}{n'^3}, \text{ or } \frac{dt}{dt'} = \frac{n^2}{n'^2}.
\]

In the second case we must suppose \( H : H' = n' : n \). [A. F.]