is derived; here we consider the function $u$ to be known, and we have $ue^{-\alpha t}$ as the particular value of $v$.

The state of the convex surface of the cylinder is subject to a condition expressed by the definite equation

$$hV + \frac{dV}{dx} = 0,$$

which must be satisfied when the radius $x$ has its total value $X$; whence we obtain the definite equation

$$h\left(1 - g \frac{X^2}{2^2} + \frac{g^2 X^4}{2^2 \cdot 4^2} - \frac{g^4 X^6}{2^2 \cdot 4^2 \cdot 6^2} + \&c.\right)$$

$$= \frac{2gX}{2^2} - \frac{4g^3 X^3}{2^2 \cdot 4^2} + \frac{6g^5 X^5}{2^2 \cdot 4^2 \cdot 6^2} - \&c. :$$

thus the number $g$ which enters into the particular value $ue^{-\alpha t}$ is not arbitrary. The number must necessarily satisfy the preceding equation, which contains $g$ and $X$.

We shall prove that this equation in $g$ in which $h$ and $X$ are given quantities has an infinite number of roots, and that all these roots are real. It follows that we can give to the variable $v$ an infinity of particular values of the form $ue^{-\alpha t}$, which differ only by the exponent $g$. We can then compose a more general value, by adding all these particular values multiplied by arbitrary coefficients. This integral which serves to resolve the proposed equation in all its extent is given by the following equation

$$v = a_1 u_1 e^{-g_1 t} + a_2 u_2 e^{-g_2 t} + a_3 u_3 e^{-g_3 t} + \&c.,$$

$g_1, g_2, g_3, \&c.$ denote all the values of $g$ which satisfy the definite equation; $u_1, u_2, u_3, \&c.$ denote the values of $u$ which correspond to these different roots; $a_1, a_2, a_3, \&c.$ are arbitrary coefficients which can only be determined by the initial state of the solid.

307. We must now examine the nature of the definite equation which gives the values of $g$, and prove that all the roots of this equation are real, an investigation which requires attentive examination.
In the series
\[1 - g \frac{X^2}{2^4} + g^4 \frac{X^4}{2^8} + g^8 \frac{X^8}{2^{12}} + \text{&c.},\]
which expresses the value which \( u \) receives when \( x = X \), we shall replace \( g X^2 \) by the quantity \( \theta \), and denoting this function of \( \theta \) by \( f(\theta) \) or \( y \), we have
\[y = f(\theta) = 1 - \theta + \frac{\theta^3}{2^2} - \frac{\theta^5}{2^3 \cdot 3^2} + \frac{\theta^7}{2^4 \cdot 3^3 \cdot 4^2} + \text{&c.},\]
the definite equation becomes
\[\frac{hX}{2} = \frac{\theta - 2 \frac{\theta^3}{2^2} + 3 \frac{\theta^5}{2^3 \cdot 3^2} - 4 \frac{\theta^7}{2^4 \cdot 3^3 \cdot 4^2} + \text{&c.}}{1 - \theta + \frac{\theta^3}{2^2} - \frac{\theta^5}{2^3 \cdot 3^2} + \frac{\theta^7}{2^4 \cdot 3^3 \cdot 4^2} - \text{&c.}},\]
or
\[\frac{hX}{2} + \theta \frac{f'(\theta)}{f(\theta)} = 0,\]
where \( f'(\theta) \) denotes the function \( \frac{df(\theta)}{d\theta} \).

Each value of \( \theta \) furnishes a value for \( g \), by means of the equation
\[g \frac{X^2}{2^4} = \theta;\]
and we thus obtain the quantities \( g_1, g_2, g_3, \&c. \) which enter in infinite number into the solution required.

The problem is then to prove that the equation
\[\frac{hX}{2} + \theta \frac{f'(\theta)}{f(\theta)} = 0\]
must have all its roots real. We shall prove in fact that the equation \( f(\theta) = 0 \) has all its roots real, that the same is the case consequently with the equation \( f'(\theta) = 0 \), and that it follows that the equation
\[A = \frac{\theta f'(\theta)}{f(\theta)}\]
has also all its roots real, \( A \) representing the known number
\[-\frac{hX}{2}.\]
308. The equation
\[ y = 1 - \theta + \frac{\theta^2}{2^2} - \frac{\theta^3}{2^2 \cdot 3^2} + \frac{\theta^4}{2^2 \cdot 3^2 \cdot 4^2} - \ldots \]
on being differentiated twice, gives the following relation
\[ y + \frac{dy}{d\theta} + \theta \frac{d^2y}{d\theta^2} = 0. \]

We write, as follows, this equation and all those which may be derived from it by differentiation,
\[ y + \frac{dy}{d\theta} + \theta \frac{d^2y}{d\theta^2} = 0, \]
\[ \frac{dy}{d\theta} + 2 \theta \frac{d^2y}{d\theta^2} + \theta \frac{d^3y}{d\theta^3} = 0, \]
\[ \frac{d^2y}{d\theta^2} + 3 \theta \frac{d^3y}{d\theta^3} + \theta \frac{d^4y}{d\theta^4} = 0, \]
\[ \ldots \]
and in general
\[ \frac{d^iy}{d\theta^i} + (i + 1) \frac{d^{i+1}y}{d\theta^{i+1}} + \theta \frac{d^{i+1}y}{d\theta^{i+1}} = 0. \]

Now if we write in the following order the algebraic equation
\[ X = 0, \]
and all those which may be derived from it by differentiation,
\[ X = 0, \quad \frac{dX}{dx} = 0, \quad \frac{d^2X}{dx^2} = 0, \quad \frac{d^3X}{dx^3} = 0, \quad \ldots \]
and if we suppose that every real root of any one of these equations on being substituted in that which precedes and in that which follows it gives two results of opposite sign; it is certain that the proposed equation \( X = 0 \) has all its roots real, and that consequently the same is the case in all the subordinate equations
\[ \frac{dX}{dx} = 0, \quad \frac{d^2X}{dx^2} = 0, \quad \frac{d^3X}{dx^3} = 0, \quad \ldots \]

These propositions are founded on the theory of algebraic equations, and have been proved long since. It is sufficient to prove that the equations
\[ y = 0, \quad \frac{dy}{d\theta} = 0, \quad \frac{d^2y}{d\theta^2} = 0, \quad \ldots \]
fulfil the preceding condition. Now this follows from the general equation
\[
\frac{dy}{d\theta} + (i + 1) \frac{d^{i+1}y}{d\theta^{i+2}} + \theta \frac{d^{i+2}y}{d\theta^{i+2}} = 0:
\]

for if we give to \( \theta \) a positive value which makes the fluxion \( \frac{d^{i+1}y}{d\theta^{i+1}} \) vanish, the other two terms \( \frac{dy}{d\theta} \) and \( \frac{d^{i+1}y}{d\theta^{i+2}} \) receive values of opposite sign. With respect to the negative values of \( \theta \) it is evident, from the nature of the function \( f'(\theta) \), that no negative value substituted for \( \theta \) can reduce to nothing, either that function, or any of the others which are derived from it by differentiation: for the substitution of any negative quantity gives the same sign to all the terms. Hence we are assured that the equation \( y = 0 \) has all its roots real and positive.

309. It follows from this that the equation \( f'(\theta) = 0 \) or \( y' = 0 \) also has all its roots real; which is a known consequence from the principles of algebra. Let us examine now what are the successive values which the term \( \theta f'(\theta) \) or \( \theta \frac{y'}{y} \) receives when we give to \( \theta \) values which continually increase from \( \theta = 0 \) to \( \theta = \infty \). If a value of \( \theta \) makes \( y' \) nothing, the quantity \( \theta \frac{y'}{y} \) becomes nothing also; it becomes infinite when \( \theta \) makes \( y \) nothing. Now it follows from the theory of equations that in the case in question, every root of \( y' = 0 \) lies between two consecutive roots of \( y = 0 \), and reciprocally. Hence denoting by \( \theta_1 \) and \( \theta_2 \) two consecutive roots of the equation \( y' = 0 \), and by \( \theta_3 \) that root of the equation \( y = 0 \) which lies between \( \theta_1 \) and \( \theta_2 \), every value of \( \theta \) included between \( \theta_1 \) and \( \theta_2 \) gives to \( y \) a sign different from that which the function \( y \) would receive if \( \theta \) had a value included between \( \theta_2 \) and \( \theta_3 \). Thus the quantity \( \theta \frac{y'}{y} \) is nothing when \( \theta = \theta_1 \); it is infinite when \( \theta = \theta_2 \), and nothing when \( \theta = \theta_3 \). The quantity \( \theta \frac{y'}{y} \) must therefore necessarily take all possible values, from \( \theta \) to infinity, in the interval from \( \theta \) to \( \theta_2 \), and must also take all possible values of the opposite sign, from infinity to zero, in the interval from \( \theta_2 \) to \( \theta_3 \). Hence the equation \( A = \theta \frac{y'}{y} \) necessarily has one
real root between \( \theta_1 \) and \( \theta_2 \) and since the equation \( y' = 0 \) has all its roots real in infinite number, it follows that the equation \( A = \frac{y'}{y} \) has the same property. In this manner we have achieved the proof that the definite equation

\[
\frac{hX}{2} = \frac{gX^2}{2^2} - 2 \frac{\varphi^2 X^4}{2^3 4^2} + 3 \frac{\varphi^3 X^6}{2^4 4^2 6^2} - \text{&c.}
\]

in which the unknown is \( g \), has all its roots real and positive. We proceed to continue the investigation of the function \( u \) and of the differential equation which it satisfies.

310. From the equation \( y + \frac{dy}{d\theta} + \theta \frac{d^2 y}{d\theta^2} = 0 \), we derive the general equation \( \frac{d^2 y}{d\theta^2} + (i + 1) \frac{d^{i+1} y}{d\theta^{i+1}} + \theta \frac{d^{i+2} y}{d\theta^{i+2}} = 0 \), and if we suppose \( \theta = 0 \) we have the equation

\[
\frac{d^{i+1} y}{d\theta^{i+1}} = - \frac{1}{i + 1} \frac{d^i y}{d\theta^i},
\]

which serves to determine the coefficients of the different terms of the development of the function \( f(\theta) \), since these coefficients depend on the values which the differential coefficients receive when the variable in them is made to vanish. Supposing the first term to be known and to be equal to 1, we have the series

\[
y = 1 - \theta + \frac{\theta^2}{2^2} - \frac{\theta^3}{2^3 3^3} + \frac{\theta^4}{2^4 3^3 4^4} - \text{&c.}
\]

If now in the equation proposed

\[
g u + \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0
\]

we make \( g \frac{x^2}{2^2} = \theta \), and seek for the new equation in \( u \) and \( \theta \), regarding \( u \) as a function of \( \theta \), we find

\[
u + \frac{du}{d\theta} + \theta \frac{d^2 u}{d\theta^2} = 0.
\]
Whence we conclude
\[ u = 1 - \theta + \frac{\theta^3}{2^2} - \frac{\theta^3}{2^2 \cdot 3^2} + \frac{\theta^4}{2^2 \cdot 3^2 \cdot 4^2} + \&c., \]
or
\[ u = 1 - \frac{g^3 x^2}{2^2} + \frac{g^3 x^4}{2^2 \cdot 4^2} - \frac{g^3 x^6}{2^3 \cdot 4^2 \cdot 6^2} + \&c. \]

It is easy to express the sum of this series. To obtain the result, develope as follows the function \( \cos (\alpha \sin x) \) in cosines of multiple arcs. We have by known transformations

\[ 2 \cos (\alpha \sin x) = e^{\frac{1}{2} \alpha x \sqrt{1}} e^{-\frac{1}{2} \alpha x \sqrt{1}} + e^{\frac{1}{2} \alpha x \sqrt{1}} e^{-\frac{1}{2} \alpha x \sqrt{1}}, \]

and denoting \( e^{x \sqrt{1}} \) by \( \omega \),

\[ 2 \cos (\alpha \sin x) = e^x - \frac{\omega}{x} \epsilon + e^{-\frac{\omega}{x}} e^x \epsilon. \]

Developing the second member according to powers of \( \omega \), we find the term which does not contain \( \omega \) in the development of \( 2 \cos (\alpha \sin x) \) to be

\[ 2 \left( 1 - \frac{\alpha^2}{2^2} + \frac{\alpha^4}{2^2 \cdot 4^2} - \frac{\alpha^6}{2^2 \cdot 4^2 \cdot 6^2} + \&c. \right). \]

The coefficients of \( \omega^1, \omega^3, \omega^5, \&c. \) are nothing, the same is the case with the coefficients of the terms which contain \( \omega^{-1}, \omega^{-3}, \omega^{-5}, \&c. \); the coefficient of \( \omega^2 \) is the same as that of \( \omega^3 \); the coefficient of \( \omega^4 \) is

\[ 2 \left( \frac{\alpha^4}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{\alpha^6}{2^2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} + \&c. \right); \]

the coefficient of \( \omega^{-4} \) is the same as that of \( \omega^4 \). It is easy to express the law according to which the coefficients succeed; but without stating it, let us write \( 2 \cos 2x \) instead of \( (\omega^2 + \omega^{-2}) \), or \( 2 \cos 4x \) instead of \( (\omega^4 + \omega^{-4}) \), and so on: hence the quantity \( 2 \cos (\alpha \sin x) \) is easily developed in a series of the form

\[ A + B \cos 2x + C \cos 4x + D \cos 6x + \&c., \]

and the first coefficient \( A \) is equal to

\[ 2 \left( 1 - \frac{\alpha^2}{2^2} + \frac{\alpha^4}{2^2 \cdot 4^2} - \frac{\alpha^6}{2^2 \cdot 4^2 \cdot 6^2} + \&c. \right); \]

if we now compare the general equation which we gave formerly

\[ \frac{1}{2} \pi \phi(x) = \frac{1}{2} \int \phi(x) dx + \cos x \int \phi(x) \cos xdx + \&c. \]
with the equation

\[ 2 \cos (\alpha \sin x) = A + B \cos 2x + C \cos 4x + \&c., \]

we shall find the values of the coefficients \( A, B, C \) expressed by definite integrals. It is sufficient here to find that of the first coefficient \( A \). We have then

\[ \frac{1}{2} A = \frac{1}{\pi} \int \cos (\alpha \sin x) \, dx, \]

the integral should be taken from \( x = 0 \) to \( x = \pi \). Hence the value of the series \( 1 - \frac{\alpha^2}{2^2} + \frac{\alpha^4}{4^2} - \frac{\alpha^6}{6^2} + \&c. \) is that of the definite integral \( \int_0^\pi dx \cos (\alpha \sin x) \). We should find in the same manner by comparison of two equations the values of the successive coefficients \( B, C, \&c. \); we have indicated these results because they are useful in other researches which depend on the same theory. It follows from this that the particular value of \( u \) which satisfies the equation

\[ gu + \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0 \]

the integral being taken from \( r = 0 \) to \( r = \pi \). Denoting by \( q \) this value of \( u \), and making \( u = qS \), we find \( S = a + b \int \frac{dx}{xq^3} \), and we have as the complete integral of the equation \( gu + \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dr} = 0 \),

\[ u = \left[ a + b \int \frac{dx}{x \left\{ \int \cos (x \sqrt{g} \sin r) \, dr \right\}^2} \right] \cos (x \sqrt{g} \sin r) \, dr. \]

\( a \) and \( b \) are arbitrary constants. If we suppose \( b = 0 \), we have, as formerly,

\[ u = \int \cos (x \sqrt{g} \sin r) \, dr. \]

With respect to this expression we add the following remarks.

311. The equation

\[ \frac{1}{\pi} \int_0^\pi \cos (\theta \sin u) \, du = 1 - \frac{\theta^2}{2^2} + \frac{\theta^4}{4^2} - \frac{\theta^6}{6^2} + \&c. \]
The integral of \( \cos (\theta \sin u) \) verifies itself. We have in fact
\[
\int \cos (\theta \sin u) \, du = \int du \left( 1 - \frac{\theta^2 \sin^2 u}{2} + \frac{\theta^4 \sin^4 u}{4} - \frac{\theta^6 \sin^6 u}{6} + \&c. \right);
\]
and integrating from \( u = 0 \) to \( u = \pi \), denoting by \( S_2, S_4, S_6, \&c. \) the definite integrals
\[
\int \sin^2 u \, du, \int \sin^4 u \, du, \int \sin^6 u \, du, \&c.,
\]
we have
\[
\int \cos (\theta \sin u) \, du = \pi - \frac{\theta^2}{2} S_2 + \frac{\theta^4}{4} S_4 - \frac{\theta^6}{6} S_6 + \&c.,
\]
it remains to determine \( S_2, S_4, S_6, \&c. \). The term \( \sin^n u \), \( n \) being an even number, may be developed thus
\[
\sin^n u = A_n + B_n \cos 2u + C_n \cos 4u + \&c.
\]
Multiplying by \( du \) and integrating between the limits \( u = 0 \) and \( u = \pi \), we have simply
\[
\int \sin^n u \, du = A_n \pi, \text{ the other terms vanish.}
\]
From the known formula for the development of the integral powers of sines, we have
\[
A_2 = \frac{1}{2^2} \cdot \frac{2}{1}, \quad A_4 = \frac{1}{2^4} \cdot \frac{3 \cdot 4}{1 \cdot 2}, \quad A_6 = \frac{1}{2^6} \cdot \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}, \&c.
\]
Substituting these values of \( S_2, S_4, S_6, \&c. \), we find
\[
\frac{1}{\pi} \int \cos (\theta \sin u) \, du = 1 - \frac{\theta^2}{2^2} + \frac{\theta^4}{2^4 \cdot 4^2} - \frac{\theta^6}{2^6 \cdot 4^4 \cdot 6^3} + \&c.
\]
We can make this result more general by taking, instead of \( \cos (\theta \sin u) \), any function whatever \( \phi \) of \( t \sin u \).

Suppose then that we have a function \( \phi (z) \) which may be developed thus
\[
\phi (z) = \phi + z \phi' + \frac{z^2}{2} \phi'' + \frac{z^3}{3} \phi''' + \&c.;
\]
we shall have
\[
\phi (t \sin u) = \phi + t \phi' \sin u + \frac{t^2}{2} \phi'' \sin^2 u + \frac{t^3}{3} \phi''' \sin^3 u + \&c.
\]
and
\[
\frac{1}{\pi} \int du \phi (t \sin u) = \phi + t S_1 \phi' + \frac{t^2}{2} S_2 \phi'' + \frac{t^3}{3} S_3 \phi''' + \&c. \ldots (e).
\]
Now, it is easy to see that the values of $S_3, S_5, S_7, \&c.$ are nothing. With respect to $S_3, S_5, S_7, \&c.$ their values are the quantities which we previously denoted by $A_3, A_5, A_7, \&c.$ For this reason, substituting these values in the equation (e) we have generally, whatever the function $\phi$ may be,

$$\frac{1}{\pi} \int \phi (t \sin u) \, du = \phi + \frac{t^2}{2} \phi'' + \frac{t^4}{4^2} \phi^{iv} + \frac{t^6}{6^2} \phi'''' + \&c.,$$

in the case in question, the function $\phi (z)$ represents $\cos z$, and we have $\phi = 1, \phi'' = -1, \phi^{iv} = 1, \phi'''' = -1$, and so on.

312. To ascertain completely the nature of the function $f (\theta)$, and of the equation which gives the values of $g$, it would be necessary to consider the form of the line whose equation is

$$y = 1 - \theta + \frac{\theta^2}{2^2} - \frac{\theta^4}{2^2 \cdot 3^2} + \&c.,$$

which forms with the axis of abscissae areas alternately positive and negative which cancel each other; the preceding remarks, also, on the expression of the values of series by means of definite integrals, might be made more general. When a function of the variable $x$ is developed according to powers of $x$, it is easy to deduce the function which would represent the same series, if the powers $x, x^2, x^3, \&c.$ were replaced by $\cos x, \cos 2x, \cos 3x, \&c.$ By making use of this reduction and of the process employed in the second paragraph of Article 235, we obtain the definite integrals which are equivalent to given series; but we could not enter upon this investigation, without departing too far from our main object.

It is sufficient to have indicated the methods which have enabled us to express the values of series by definite integrals.

We will add only the development of the quantity $\frac{f'(\theta)}{f(\theta)}$ in a continued fraction.

313. The undetermined $y$ or $f(\theta)$ satisfies the equation

$$y + \frac{dy}{d\theta} + \theta \frac{d^2 y}{d\theta^2} = 0,$$
whence we derive, denoting the functions
\[ \frac{dy}{d\theta}, \frac{d^2y}{d\theta^2}, \frac{d^3y}{d\theta^3}, \text{ &c.} \]
by \( y', y'', y''', \text{ &c.} \),

\[ -y = y' + \theta y'' \quad \text{or} \quad \frac{y'}{y} = \frac{-y'}{y'} \quad = \quad \frac{-1}{1 + \theta \frac{y''}{y}} , \]

\[ -y' = 2y'' + \theta y''' , \quad \frac{y''}{y} = \frac{-y''}{2y'' + \theta y'''} = \frac{-1}{2 + \theta \frac{y'''}{y''}} , \]

\[ -y'' = 3y''' + \theta y'''' , \quad \frac{y'''}{y'} = \frac{-y''}{3y''' + \theta y'''} = \frac{-1}{3 + \theta \frac{y''''}{y'''}} , \]

\&c.;

whence we conclude

\[ \frac{y'}{y} = \frac{-1}{1 - \frac{\theta}{2} - \frac{\theta}{3} - \frac{\theta}{4} - \frac{\theta}{5} - \&c.} . \]

Thus the value of the function \( -\frac{\theta f'(\theta)}{f(\theta)} \) which enters into the definite equation, when expressed as an infinite continued fraction, is

\[ \frac{\theta}{1 - \frac{\theta}{2} - \frac{\theta}{3} - \frac{\theta}{4} - \frac{\theta}{5} - \&c.} . \]

314. We shall now state the results at which we have up to this point arrived.

If the variable radius of the cylindrical layer be denoted by \( x \), and the temperature of the layer by \( v \), a function of \( x \) and the time \( t \); the required function \( v \) must satisfy the partial differential equation

\[ \frac{dv}{dt} = k \left( \frac{d^2v}{dx^2} + \frac{1}{x} \frac{dv}{dx} \right) ; \]

for \( v \) we may assume the following value

\[ v = ue^{-mt} ; \]

\( u \) is a function of \( x \), which satisfies the equation

\[ \frac{m}{k} u + \frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0 . \]
If we make \( \theta = \frac{m x^2}{k 2^2} \), and consider \( u \) as a function of \( x \), we have

\[
u + \frac{du}{d\theta} + \theta \frac{d^2 u}{d\theta^2} = 0.
\]

The following value

\[
u = 1 - \theta + \frac{\theta^2}{2^2} - \frac{\theta^3}{2^2 \cdot 3^2} + \frac{\theta^4}{2^2 \cdot 3^2 \cdot 4^2} - \&c.
\]

satisfies the equation in \( u \) and \( \theta \). We therefore assume the value of \( u \) in terms of \( x \) to be

\[
u = 1 - \frac{m x^2}{k 2^2} + \frac{m^2 x^2}{k^2 2^2} - \frac{m^3 x^2}{k^3 2^2 \cdot 4^2} - \frac{m^4 x^2}{k^3 \cdot 3^2 \cdot 4^2} - \&c.,
\]

the sum of this series is

\[
\frac{1}{\pi} \int \cos \left( x \sqrt{\frac{m}{k}} \sin r \right) dr;
\]

the integral being taken from \( r = 0 \) to \( r = \pi \). This value of \( \nu \) in terms of \( x \) and \( m \) satisfies the differential equation, and retains a finite value when \( x \) is nothing. Further, the equation \( hu + \frac{du}{dx} = 0 \) must be satisfied when \( x = X \) the radius of the cylinder. This condition would not hold if we assigned to the quantity \( m \) any value whatever; we must necessarily have the equation

\[
\frac{hX}{2} = \theta \frac{\theta}{1 - 2 - 3 - 4 - 5 - \&c.},
\]

in which \( \theta \) denotes \( \frac{m X^2}{k 2^2} \).

This definite equation, which is equivalent to the following,

\[
\frac{hX}{2} \left( 1 - \theta + \frac{\theta^2}{2^2} - \frac{\theta^3}{2^2 \cdot 3^2} + \&c. \right) = \theta - \frac{2\theta^2}{2^2} + \frac{3\theta^3}{2^2 \cdot 3^2} - \&c.,
\]

gives to \( \theta \) an infinity of real values denoted by \( \theta_1, \theta_2, \theta_3, \&c. \); the corresponding values of \( m \) are

\[
\frac{2^2 k \theta_1}{X^2}, \frac{2^2 k \theta_2}{X^2}, \frac{2^2 k \theta_3}{X^2}, \&c.;
\]

thus a particular value of \( \nu \) is expressed by

\[
\pi \nu = e^{-\frac{2^2 k \theta_1}{X^2}} \int \cos \left( \frac{2 x}{X} \sqrt{\theta_1 \sin q} \right) dq.
\]
We can write, instead of $\theta$, one of the roots $\theta, \theta_2, \theta_3, \&c.$ and compose by means of them a more general value expressed by the equation

$$\pi \nu = a_1 e^{-\frac{2\pi k n_1}{x^2}} \int \cos \left(2 \frac{x}{X} \sqrt{\theta} \sin q \right) dq + a_2 e^{-\frac{2\pi k n_2}{x^2}} \int \cos \left(2 \frac{x}{X} \sqrt{\theta} \sin q \right) dq + a_3 e^{-\frac{2\pi k n_3}{x^2}} \int \cos \left(2 \frac{x}{X} \sqrt{\theta} \sin q \right) dq + \&c.$$

$a_1, a_2, a_3, \&c.$ are arbitrary coefficients; the variable $q$ disappears after the integrations, which should be taken from $q = 0$ to $q = \pi$.

315. To prove that this value of $\nu$ satisfies all the conditions of the problem and contains the general solution, it remains only to determine the coefficients $a_1, a_2, a_3, \&c.$ from the initial state. Take the equation

$$\nu = a_1 e^{-m_1 t} u_1 + a_2 e^{-m_2 t} u_2 + a_3 e^{-m_3 t} u_3 + \&c.,$$

in which $u_1, u_2, u_3, \&c.$ are the different values assumed by the function $u$, or

$$1 - \frac{m}{k} \frac{x^2}{2^2} + \frac{m^2}{k^2} \frac{x^4}{4^2} - \&c.$$ when, instead of $\frac{m}{k}$, the values $g_1, g_2, g_3, \&c.$ are successively substituted. Making in it $t = 0$, we have the equation

$$V = a_1 u_1 + a_2 u_2 + a_3 u_3 + \&c.,$$
in which $V$ is a given function of $x$. Let $\phi (x)$ be this function; if we represent the function $u_i$ whose index is $i$ by $\psi (x \sqrt{g_i})$, we have

$$\phi (x) = a_1 \psi (x \sqrt{g_1}) + a_2 \psi (x \sqrt{g_2}) + a_3 \psi (x \sqrt{g_3}) + \&c.$$

To determine the first coefficient, multiply each member of the equation by $\sigma, dx, \sigma_1$ being a function of $x$, and integrate from $x = 0$ to $x = X$. We then determine the function $\sigma_1$, so that after the integrations the second member may reduce to the first term only, and the coefficient $a_1$ may be found, all the other integrals
having null values. Similarly to determine the second coefficient $a_2$, we multiply both terms of the equation

$$\phi(x) = a_2 u_1 + a_2 u_2 + a_3 u_3 + \&c.$$ 

by another factor $\sigma_2 \, dx$, and integrate from $x = 0$ to $x = X$. The factor $\sigma_2$ must be such that all the integrals of the second member vanish, except one, namely that which is affected by the coefficient $a_2$. In general, we employ a series of functions of $x$ denoted by $\sigma_1, \sigma_2, \sigma_3, \&c.$ which correspond to the functions $u_1, u_2, u_3, \&c.$; each of the factors $\sigma$ has the property of making all the terms which contain definite integrals disappear in integration except one; in this manner we obtain the value of each of the coefficients $a_1, a_2, a_3, \&c.$ We must now examine what functions enjoy the property in question.

316. Each of the terms of the second member of the equation is a definite integral of the form $\int \sigma u \, dx$; $u$ being a function of $x$ which satisfies the equation

$$\frac{m}{k} u + \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0;$$

we have therefore $\int \sigma u \, dx = - \frac{k}{m} \int \left( \frac{du}{x \, dx} + \sigma \frac{d^2 u}{dx} \right)$.

Developing, by the method of integration by parts, the terms

$$\int \frac{\sigma}{x \, dx} \, dx \text{ and } \int \frac{\sigma}{x} \, dx,$$

we have

$$\int \frac{\sigma}{x \, dx} \, dx = C + u \frac{\sigma}{x} - \int u \left( \frac{\sigma}{x} \right)$$

and

$$\int \frac{\sigma}{x} \, dx = D + \frac{du}{dx} \sigma - u \frac{d\sigma}{dx} + \int u \frac{d\sigma}{dx} \, dx.$$

The integrals must be taken between the limits $x = 0$ and $x = X$, by this condition we determine the quantities which enter into the development, and are not under the integral signs. To indicate that we suppose $x = 0$ in any expression in $x$, we shall affect that expression with the suffix $\alpha$; and we shall give it the suffix $\omega$ to indicate the value which the function of $x$ takes, when we give to the variable $x$ its last value $X$. 

Supposing \( x = 0 \) in the two preceding equations we have
\[
0 = C + \left( \frac{u}{x} \frac{\sigma}{\alpha} \right) \quad \text{and} \quad 0 = D + \left( \frac{\frac{du}{dx} \sigma - u \frac{d\sigma}{dx}}{\alpha} \right),
\]
thus we determine the constants \( C \) and \( D \). Making then \( x = X \) in the same equations, and supposing the integral to be taken from \( x = 0 \) to \( x = X \), we have
\[
\int \frac{\sigma}{x} \frac{du}{dx} \, dx = \left( \frac{u}{x} \frac{\sigma}{\alpha} \right) - \left( \frac{u}{x} \frac{\sigma}{\alpha} \right) - \int u \frac{\sigma}{x} \, dx
\]
and
\[
\int \frac{d^2u}{dx^2} \, dx = \left( \frac{\frac{du}{dx} \sigma - u \frac{d\sigma}{dx}}{\alpha} \right) - \left( \frac{\frac{du}{dx} \sigma - u \frac{d\sigma}{dx}}{\alpha} \right) + \int u \frac{d^2\sigma}{dx^2} \, dx,
\]
thus we obtain the equation
\[
- \frac{m}{k} \int \sigma \, u \, dx = \int \left\{ u \frac{d^2\sigma}{dx^2} - u \frac{\frac{d\sigma}{dx}}{x} \right\} \, dx + \left( \frac{\frac{du}{dx} \sigma - u \frac{d\sigma}{dx} + u \frac{\sigma}{x}}{\alpha} \right) - \left( \frac{\frac{du}{dx} \sigma - u \frac{d\sigma}{dx} + u \frac{\sigma}{x}}{\alpha} \right).
\]

317. If the quantity \( \frac{d^2\sigma}{dx^2} - \frac{d\sigma}{dx} \frac{\frac{\sigma}{x}}{dx} \) which multiplies \( u \) under the sign of integration in the second member were equal to the product of \( \sigma \) by a constant coefficient, the terms
\[
\int \left\{ u \frac{d^2\sigma}{dx^2} - u \frac{\frac{d\sigma}{dx}}{dx} \right\} \, dx \quad \text{and} \quad \int \sigma \, u \, dx
\]
would be collected into one, and we should obtain for the required integral \( \int \sigma \, u \, dx \) a value which would contain only determined quantities, with no sign of integration. It remains only to equate that value to zero.

Suppose then the factor \( \sigma \) to satisfy the differential equation of the second order \( \frac{n}{k} \sigma + \frac{d^2\sigma}{dx^2} \frac{\frac{\sigma}{x}}{dx} = 0 \) in the same manner as the function \( u \) satisfies the equation
\[
\frac{m}{k} u + \frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0,
\]
F. H.
m and n being constant coefficients, we have

$$\frac{n-m}{k} \int \sigma u dx = \left( \frac{du}{dx} \sigma - u \frac{d\sigma}{dx} + u \frac{\sigma}{x} \right) - \left( \frac{du}{dx} \sigma - u \frac{d\sigma}{dx} + u \frac{\sigma}{x} \right) \xi.$$ 

Between \( u \) and \( \sigma \) a very simple relation exists, which is discovered when in the equation

$$\frac{n}{k} \sigma + \frac{d^2 \sigma}{dx^2} - \frac{d (\sigma)}{dx} = 0 \quad \text{we suppose} \quad \sigma = xs; \quad \text{as the result of this substitution we have the equation}$$

$$\frac{n}{k} s + \frac{d^2 s}{dx^2} + \frac{1}{x} \frac{ds}{dx} = 0,$$

which shews that the function \( s \) depends on the function \( u \) given by the equation

$$\frac{m}{k} u + \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0.$$

To find \( s \) it is sufficient to change \( m \) into \( n \) in the value of \( u \); the value of \( u \) has been denoted by \( \psi \left( x \sqrt{\frac{n}{k}} \right) \), that of \( \sigma \) will therefore be \( x \psi \left( x \sqrt{\frac{n}{k}} \right) \).

We have then

$$\frac{du}{dx} \sigma - u \frac{d\sigma}{dx} + u \frac{\sigma}{x} = \psi \left( x \sqrt{\frac{m}{k}} \right) \psi \left( x \sqrt{\frac{n}{k}} \right) - \psi \left( x \sqrt{\frac{n}{k}} \right) \psi \left( x \sqrt{\frac{m}{k}} \right) - \psi \left( x \sqrt{\frac{m}{k}} \right) \psi \left( x \sqrt{\frac{n}{k}} \right) \psi \left( x \sqrt{\frac{n}{k}} \right).$$

the two last terms destroy each other, it follows that on making \( x = 0 \), which corresponds to the suffix \( \alpha \), the second member vanishes completely. We conclude from this the following equation

$$\frac{n-m}{k} \int \sigma u dx = X \sqrt{\frac{m}{k}} \psi \left( X \sqrt{\frac{m}{k}} \right) \psi \left( X \sqrt{\frac{n}{k}} \right)$$

$$- X \sqrt{\frac{n}{k}} \psi \left( X \sqrt{\frac{n}{k}} \right) \psi \left( X \sqrt{\frac{m}{k}} \right) \ldots \ldots \ldots (f).$$
It is easy to see that the second member of this equation is always nothing when the quantities \( m \) and \( n \) are selected from those which we formerly denoted by \( m_1, m_2, m_3, \&c. \).

We have in fact

\[
hX = -X \sqrt{\frac{m}{k}} \frac{\psi'(X \sqrt{\frac{m}{k}})}{\psi(X \sqrt{\frac{m}{k}})} \quad \text{and} \quad hX = -X \sqrt{\frac{n}{k}} \frac{\psi'(X \sqrt{\frac{n}{k}})}{\psi(X \sqrt{\frac{n}{k}})},
\]

comparing the values of \( hX \) we see that the second member of the equation \((f)\) vanishes.

It follows from this that after we have multiplied by \( \sigma dx \) the two terms of the equation

\[
\phi(x) = a_1 u_1 + a_2 u_2 + a_3 u_3 + \&c.,
\]

and integrated each side from \( x = 0 \) to \( x = X \), in order that each of the terms of the second member may vanish, it suffices to take for \( \sigma \) the quantity \( xu \) or \( x\psi(x \sqrt{\frac{m}{k}}) \).

We must except only the case in which \( n = m \), when the value of \( \int \sigma u dx \) derived from the equation \((f)\) is reduced to the form \( \frac{0}{0} \), and is determined by known rules.

318. If \( \sqrt{\frac{m}{k}} = \mu \) and \( \sqrt{\frac{n}{k}} = \nu \), we have

\[
\int x\psi(\mu) \psi'(\nu) \, dx = \frac{\mu X \psi'(\mu X) \psi(\nu X) - \nu X \psi'(\nu X) \psi(\mu X)}{\nu^2 - \mu^2}.
\]

If the numerator and denominator of the second member are separately differentiated with respect to \( \nu \), the factor becomes, on making \( \mu = \nu \),

\[
\frac{\mu X^2 \psi'' - X \psi \psi' - \mu X^2 \psi \psi''}{2\mu}.
\]

We have on the other hand the equation

\[
\mu^2 u + \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} = 0, \quad \text{or} \quad \mu^2 \psi' + \frac{\mu}{x} \psi' + \mu \psi'' = 0,
\]

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and also

\[ hx \psi + \mu x \psi' = 0, \]

or,

\[ h \psi + \mu \psi' = 0; \]

hence we have

\[ \left( \mu^2 - \frac{h}{x} \right) \psi + \mu^2 \psi'' = 0, \]

we can therefore eliminate the quantities \( \psi' \) and \( \psi'' \) from the integral which is required to be evaluated, and we shall find as the value of the integral sought

\[ \frac{1}{2} X^2 \psi^2 \left( \frac{\mu^2 + h^2}{\mu^2} \right) \text{ or } \frac{X^2 u^2}{2} \left( 1 + \frac{kh}{m_i} \right), \]

putting for \( \mu \) its value, and denoting by \( U_i \) the value which the function \( u \) or \( \psi \left( x \sqrt{\frac{m_i}{k}} \right) \) takes when we suppose \( x = X \). The index \( i \) denotes the order of the root \( m \) of the definite equation which gives an infinity of values of \( m \). If we substitute \( m_i \) or \( \frac{2^k h \theta_i}{X^2} \text{ in } \frac{X^2 U_i^2}{2} \left( 1 + \frac{k h^2}{m_i} \right) \), we have

\[ \frac{1}{2} X^2 U_i^2 \left\{ 1 + \left( \frac{hX}{2 \sqrt{\theta_i}} \right)^2 \right\}. \]

319. It follows from the foregoing analysis that we have the two equations

\[ \int_0^X xu \psi_i \, dx = 0 \text{ and } \int_0^X xu \psi_i^2 \, dx = \left\{ 1 + \left( \frac{hX}{2 \sqrt{\theta_i}} \right)^2 \right\} \frac{X^2 U_i^2}{2}, \]

the first holds whenever the number \( i \) and \( j \) are different, and the second when these numbers are equal.

Taking then the equation \( \phi (x) = a_i u_i + a_2 u_2 + a_3 u_3 + \&c. \), in which the coefficients \( a_1, a_2, a_3, \&c. \) are to be determined, we shall find the coefficient denoted by \( a_i \) by multiplying the two members of the equation by \( xu \psi_i \, dx \), and integrating from \( x = 0 \) to \( x = X \); the second member is reduced by this integration to one term only, and we have the equation

\[ 2 \int x \phi (x) u_i \, dx = a_i X^2 U_i^2 \left\{ 1 + \left( \frac{hX}{2 \sqrt{\theta_i}} \right)^2 \right\}, \]
which gives the value of $a_i$. The coefficients $a_1, a_2, a_3, \ldots a_p$ being thus determined, the condition relative to the initial state expressed by the equation $\phi(x) = a_1 u_1 + a_2 u_2 + a_3 u_3 + \&c.$, is fulfilled.

We can now give the complete solution of the proposed problem; it is expressed by the following equation:

$$vX^2 = \frac{\int_0^X x\phi(x) u_1 dx}{U_1^2 \left(1 + \frac{h^2 X^2}{2^2 \theta_1}\right)} u_1 e^{-\frac{2^{2k} t}{X^2 \theta_1}} + \frac{\int_0^X x\phi(x) u_2 dx}{U_2^2 \left(1 + \frac{h^2 X^2}{2^2 \theta_2}\right)} u_2 e^{-\frac{2^{2k} t}{X^2 \theta_2}} + \&c.$$

The function of $x$ denoted by $u$ in the preceding equation is expressed by

$$\frac{1}{2} \int \cos \left(\frac{2x}{X} \sqrt{\theta_1} \sin q\right) dq;$$

all the integrals with respect to $x$ must be taken from $x=0$ to $x=X$, and to find the function $u$ we must integrate from $q=0$ to $q=\pi$; $\phi(x)$ is the initial value of the temperature, taken in the interior of the cylinder at a distance $x$ from the axis, which function is arbitrary, and $\theta_1, \theta_2, \theta_3, \&c.$ are the real and positive roots of the equation

$$\frac{hX}{2} = \frac{\theta}{1 - 2 - \frac{\theta}{3} - \frac{\theta}{4} - \frac{\theta}{5} - \&c.}.$$

320. If we suppose the cylinder to have been immersed for an infinite time in a liquid maintained at a constant temperature, the whole mass becomes equally heated, and the function $\phi(x)$ which represents the initial state is represented by unity. After this substitution, the general equation represents exactly the gradual progress of the cooling.

If $t$ the time elapsed is infinite, the second member contains only one term, namely, that which involves the least of all the roots $\theta_1, \theta_2, \theta_3, \&c.$; for this reason, supposing the roots to be arranged according to their magnitude, and $\theta$ to be the least, the final state of the solid is expressed by the equation

$$vX^2 = \frac{\int x\phi(x) u_1 dx}{U_1^2 \left(1 + \frac{h^2 X^2}{2^2 \theta_1}\right)} u_1 e^{-\frac{2^{2k} t}{X^2 \theta_1}}.$$
From the general solution we might deduce consequences similar to those offered by the movement of heat in a spherical mass. We notice first that there are an infinite number of particular states, in each of which the ratios established between the initial temperatures are preserved up to the end of the cooling. When the initial state does not coincide with one of these simple states, it is always composed of several of them, and the ratios of the temperatures change continually, according as the time increases. In general the solid arrives very soon at the state in which the temperatures of the different layers decrease continually preserving the same ratios. When the radius \( X \) is very small\(^1\), we find that the temperatures decrease in proportion to the fraction \( e^{-\frac{2h}{CDX}} \).

If on the contrary the radius \( X \) is very large\(^2\), the exponent of \( e \) in the term which represents the final system of temperatures contains the square of the whole radius. We see by this what influence the dimension of the solid has upon the final velocity of cooling. If the temperature\(^3\) of the cylinder whose radius is \( X \), passes from the value \( A \) to the lesser value \( B \), in the time \( T \), the temperature of a second cylinder of radius equal to \( X' \) will pass from \( A \) to \( B \) in a different time \( T' \). If the two sides are thin, the ratio of the times \( T \) and \( T' \) will be that of the diameters. If, on the contrary, the diameters of the cylinders are very great, the ratio of the times \( T \) and \( T' \) will be that of the squares of the diameters.

\(^1\) When \( X \) is very small, \( \theta = \frac{hX}{2} \), from the equation in Art. 314. Hence
\[
-\frac{2ht}{X^2} \theta \quad \text{becomes} \quad e^{-\frac{2ht}{X^2}}.
\]
In the text, \( h \) is the surface conducibility.

\(^2\) When \( X \) is very large, a value of \( \theta \) nearly equal to one of the roots of the quadratic equation \( 1 = \theta - \frac{3\theta}{2} - \frac{4\theta}{5} \) will make the continued fraction in Art. 314 assume its proper magnitude. Hence \( \theta = 1.446 \) nearly, and
\[
-\frac{2ht}{X^2} \theta \quad \text{becomes} \quad e^{-\frac{2ht}{X^2}}.
\]
The least root of \( f(\theta) = 0 \) is 1.4467, neglecting terms after \( \theta^4 \).

\(^3\) The temperature intended is the mean temperature, which is equal to
\[
\frac{1}{\pi X^2} \int_{0}^{X} \nu d(\pi x^2), \quad \text{or} \quad \frac{2}{X^2} \int_{0}^{X} \nu x dx. \quad [A. F.]
\]