CHAPTER V.

OF THE PROPAGATION OF HEAT IN A SOLID SPHERE.

SECTION I.

General solution.

283. The problem of the propagation of heat in a sphere has been explained in Chapter II., Section 2, Article 117; it consists in integrating the equation

$$\frac{dv}{dt} = k \left( \frac{d^2v}{dx^2} + \frac{2}{x} \frac{dv}{dx} \right),$$

so that when $x = X$ the integral may satisfy the condition

$$\frac{dv}{dx} + hv = 0,$$

$k$ denoting the ratio $\frac{K}{CD}$, and $h$ the ratio $\frac{h}{K}$ of the two conductibilities; $v$ is the temperature which is observed after the time $t$ has elapsed in a spherical layer whose radius is $x$; $X$ is the radius of the sphere; $v$ is a function of $x$ and $t$, which is equal to $F'(x)$ when we suppose $t = 0$. The function $F'(x)$ is given, and represents the initial and arbitrary state of the solid.

If we make $y = vx$, $y$ being a new unknown, we have, after the substitutions, $\frac{dy}{dt} = k \frac{d^2y}{dx^2}$: thus we must integrate the last equation, and then take $v = \frac{y}{x}$. We shall examine, in the first place, what are the simplest values which can be attributed to $y$, and then form a general value which will satisfy at the same
time the differential equation, the condition relative to the surface, and the initial state. It is easily seen that when these three conditions are fulfilled, the solution is complete, and no other can be found.

284. Let \( y = e^{mt}u \), \( u \) being a function of \( x \), we have

\[
m_u = k \frac{d^2u}{dx^2}.
\]

First, we notice that when the value of \( t \) becomes infinite, the value of \( v \) must be nothing at all points, since the body is completely cooled. Negative values only can therefore be taken for \( m \). Now \( k \) has a positive numerical value, hence we conclude that the value of \( u \) is a circular function, which follows from the known nature of the equation

\[
m_u = k \frac{d^2u}{dx^2}.
\]

Let \( u = A \cos nx + B \sin nx \); we have the condition \( m = -kn^2 \).

Thus we can express a particular value of \( v \) by the equation

\[
v = \frac{e^{-kn^2t}}{x} (A \cos nx + B \sin nx),
\]

where \( n \) is any positive number, and \( A \) and \( B \) are constants. We may remark, first, that the constant \( A \) ought to be nothing; for the value of \( v \) which expresses the temperature at the centre, when we make \( x = 0 \), cannot be infinite; hence the term \( A \cos nx \) should be omitted.

Further, the number \( n \) cannot be taken arbitrarily. In fact, if in the definite equation \( \frac{dv}{dx} + hv = 0 \) we substitute the value of \( v \), we find

\[
nx \cos nx + (hx - 1) \sin nx = 0.
\]

As the equation ought to hold at the surface, we shall suppose in it \( x = X \) the radius of the sphere, which gives

\[
\frac{nx}{\tan nx} = 1 - hX.
\]

Let \( \lambda \) be the number \( 1 - hX \), and \( nX = \epsilon \), we have \( \frac{\epsilon}{\tan \epsilon} = \lambda \).

We must therefore find an arc \( \epsilon \), which divided by its tangent
gives a known quotient $\lambda$, and afterwards take $n = \frac{e}{X}$. It is evident that there are an infinity of such arcs, which have a given ratio to their tangent; so that the equation of condition
\[
\frac{nX}{\tan nX} = 1 - hX
\]
has an infinite number of real roots.

285. Graphical constructions are very suitable for exhibiting the nature of this equation. Let $u = \tan \epsilon$ (fig. 12), be the equation to a curve, of which the arc $\epsilon$ is the abscissa, and $u$ the ordinate; and let $u = \frac{e}{\lambda}$ be the equation to a straight line, whose co-ordinates are also denoted by $\epsilon$ and $u$. If we eliminate $u$ from these two equations, we have the proposed equation $\frac{e}{\lambda} = \tan \epsilon$. The unknown $\epsilon$ is therefore the abscissa of the point of intersection of the curve and the straight line. This curved line is composed of an infinity of arcs; all the ordinates corresponding to abscissae
\[
\frac{1}{2} \pi, \frac{3}{2} \pi, \frac{5}{2} \pi, \frac{7}{2} \pi, \text{ &c.}
\]
are infinite, and all those which correspond to the points $0, \pi, 2\pi, 3\pi, \text{ &c.}$ are nothing. To trace the straight line whose equation is $u = \frac{e}{\lambda} = \frac{e}{1 - hX}$, we form the square $\omega i \omega i$, and measuring the quantity $hX$ from $\omega$ to $h$, join the point $h$ with the origin $o$. The curve non whose equation is $u = \tan \epsilon$ has for
tangent at the origin a line which divides the right angle into two equal parts, since the ultimate ratio of the arc to the tangent is 1. We conclude from this that if \( \lambda \) or \( l - hX \) is a quantity less than unity, the straight line \( mom \) passes from the origin above the curve \( non \), and there is a point of intersection of the straight line with the first branch. It is equally clear that the same straight line cuts all the further branches \( n\pi n, n2\pi n, \) &c. Hence the equation \( \frac{e}{\tan e} = \lambda \) has an infinite number of real roots. The first is included between 0 and \( \frac{\pi}{2} \), the second between \( \pi \) and \( \frac{3\pi}{2} \), the third between \( 2\pi \) and \( \frac{5\pi}{2} \), and so on. These roots approach very near to their upper limits when they are of a very advanced order.

286. If we wish to calculate the value of one of the roots, for example, of the first, we may employ the following rule: write down the two equations \( e = \arctan u \) and \( u = \frac{e}{\lambda} \), \( \arctan u \) denoting the length of the arc whose tangent is \( u \). Then taking any number for \( u \), deduce from the first equation the value of \( e \); substitute this value in the second equation, and deduce another value of \( u \); substitute the second value of \( u \) in the first equation; thence we deduce a value of \( e \), which, by means of the second equation, gives a third value of \( u \). Substituting it in the first equation we have a new value of \( e \). Continue thus to determine \( u \) by the second equation, and \( e \) by the first. The operation gives values more and more nearly approaching to the unknown \( e \), as is evident from the following construction.

In fact, if the point \( u \) correspond (see fig. 13) to the arbitrary value which is assigned to the ordinate \( u \); and if we substitute this value in the first equation \( e = \arctan u \), the point \( e \) will correspond to the abscissa which we have calculated by means of this equation. If this abscissa \( e \) be substituted in the second equation \( u = \frac{e}{\lambda} \), we shall find an ordinate \( u' \) which corresponds to the point \( u' \). Substituting \( u' \) in the first equation, we find an abscissa \( e' \) which corresponds to the point \( e' \); this abscissa being
then substituted in the second equation gives rise to an ordinate $u'$, which when substituted in the first, gives rise to a third abscissa $e''$, and so on to infinity. That is to say, in order to represent the continued alternate employment of the two preceding equations, we must draw through the point $u$ a horizontal line up to the curve, and through $e$ the point of intersection draw a vertical as far as the straight line, through the point of intersection $u'$ draw a horizontal up to the curve, through the point of intersection $e'$ draw a vertical as far as the straight line, and so on to infinity, descending more and more towards the point sought.

287. The foregoing figure (13) represents the case in which the ordinate arbitrarily chosen for $u$ is greater than that which corresponds to the point of intersection. If, on the other hand, we chose for the initial value of $u$ a smaller quantity, and employed in the same manner the two equations $e = \arctan u$, $u = \frac{e}{\lambda}$, we should again arrive at values successively closer to the unknown value. Figure 14 shews that in this case we rise continually towards the point of intersection by passing through the points $u e u' e' u'' e''$, &c. which terminate the horizontal and vertical lines. Starting from a value of $u$ which is too small, we obtain quantities $e e' e'' e'''$, &c. which converge towards the unknown value, and are smaller than it; and starting from a value of $u$ which is too great, we obtain quantities which also converge to the unknown value, and each of which is greater than it. We therefore ascertain
successively closer limits between the which magnitude sought is always included. Either approximation is represented by the formula

$$e = \ldots \arctan \left[ \arctan \left( \frac{1}{\lambda} \arctan \left( \frac{1}{\lambda} \arctan \frac{1}{\lambda} \right) \right) \right].$$

When several of the operations indicated have been effected, the successive results differ less and less, and we have arrived at an approximate value of $e$.

288. We might attempt to apply the two equations

$$e = \arctan u \text{ and } u = \frac{e}{\lambda}$$

in a different order, giving them the form $u = \tan e$ and $e = \lambda u$. We should then take an arbitrary value of $e$, and, substituting it in the first equation, we should find a value of $u$, which being substituted in the second equation would give a second value of $e$; this new value of $e$ could then be employed in the same manner as the first. But it is easy to see, by the constructions of the figures, that in following this course of operations we depart more and more from the point of intersection instead of approaching it, as in the former case. The successive values of $e$ which we should obtain would diminish continually to zero, or would increase without limit. We should pass successively from $e''$ to $u''$, from $u''$ to $e'$, from $e'$ to $u'$, from $u'$ to $e$, and so on to infinity.

The rule which we have just explained being applicable to the calculation of each of the roots of the equation

$$\frac{e}{\tan e} = 1 - \lambda X,$$

which moreover have given limits, we must regard all these roots as known numbers. Otherwise, it was only necessary to be assured that the equation has an infinite number of real roots. We have explained this process of approximation because it is founded on a remarkable construction, which may be usefully employed in several cases, and which exhibits immediately the nature and limits of the roots; but the actual application of the process to the equation in question would be tedious; it would be easy to resort in practice to some other mode of approximation.
289. We now know a particular form which may be given to the function \( v \) so as to satisfy the two conditions of the problem. This solution is represented by the equation

\[
v = \frac{Ae^{-knt} \sin nx}{x} \quad \text{or} \quad v = ae^{-knt} \frac{\sin nx}{nx}.
\]

The coefficient \( a \) is any number whatever, and the number \( n \) is such that \( \frac{nX}{\tan nX} = 1 - kX \). It follows from this that if the initial temperatures of the different layers were proportional to the quotient \( \frac{\sin nx}{nx} \), they would all diminish together, retaining between themselves throughout the whole duration of the cooling the ratios which had been set up; and the temperature at each point would decrease as the ordinate of a logarithmic curve whose abscissa would denote the time passed. Suppose, then, the arc \( \epsilon \) being divided into equal parts and taken as abscissa, we raise at each point of division an ordinate equal to the ratio of the sine to the arc. The system of ordinates will indicate the initial temperatures, which must be assigned to the different layers, from the centre to the surface, the whole radius \( X \) being divided into equal parts. The arc \( \epsilon \) which, on this construction, represents the radius \( X \), cannot be taken arbitrarily; it is necessary that the arc and its tangent should be in a given ratio. As there are an infinite number of arcs which satisfy this condition, we might thus form an infinite number of systems of initial temperatures, which could exist of themselves in the sphere, without the ratios of the temperatures changing during the cooling.

290. It remains only to form any initial state by means of a certain number, or of an infinite number of partial states, each of which represents one of the systems of temperatures which we have recently considered, in which the ordinate varies with the distance \( x \), and is proportional to the quotient of the sine by the arc. The general movement of heat in the interior of a sphere will then be decomposed into so many particular movements, each of which is accomplished freely, as if it alone existed.

Denoting by \( n_1, n_2, n_3, \&c. \), the quantities which satisfy the equation \( \frac{nX}{\tan nX} = 1 - kX \), and supposing them to be arranged in
order, beginning with the least, we form the general equation

\[ vx = a_1 e^{-kmx} \sin nx + a_2 e^{-kmx} \sin nx + a_3 e^{-kmx} \sin nx + \&c. \]

If \( t \) be made equal to 0, we have as the expression of the initial state of temperatures

\[ vx = a_1 \sin nx + a_2 \sin nx + a_3 \sin nx + \&c. \]

The problem consists in determining the coefficients \( a_1, a_2, a_3 \&c. \), whatever be the initial state. Suppose then that we know the values of \( v \) from \( x = 0 \) to \( x = X \), and represent this system of values by \( F(x) \); we have

\[ F(x) = \frac{1}{x} (a_1 \sin nx + a_2 \sin nx + a_3 \sin nx + a_4 \sin nx + \&c.) \ldots (e). \]

291. To determine the coefficient \( a_1 \), multiply both members of the equation by \( x \sin nx \, dx \), and integrate from \( x = 0 \) to \( x = X \). The integral \( \int \sin mx \sin nx \, dx \) taken between these limits is

\[ \frac{1}{m^2 - n^2} (- m \sin nX \cos mX + n \sin mX \cos nX). \]

If \( m \) and \( n \) are numbers chosen from the roots \( n_1, n_2, n_3, \&c. \), which satisfy the equation \( \frac{nX}{\tan nX} = 1 - hX \), we have

\[ \frac{mX}{\tan mX} = \frac{nX}{\tan nX}, \]

or

\[ m \cos mX \sin nX - n \sin mX \cos nX = 0. \]

We see by this that the whole value of the integral is nothing; but a single case exists in which the integral does not vanish, namely, when \( m = n \). It then becomes \( \frac{0}{0} \); and, by application of known rules, is reduced to

\[ \frac{1}{2} X - \frac{1}{4n} \sin 2nX. \]

1 Of the possibility of representing an arbitrary function by a series of this form a demonstration has been given by Sir W. Thomson, Camb. Math. Journal, Vol. iii. pp. 25-27. [A. F.]
It follows from this that in order to obtain the value of the coefficient $a_1$, in equation (e), we must write

$$2 \int x \sin n_1 x \, F(x) \, dx = a_1 \left( X - \frac{1}{2n_1} \sin 2n_1 X \right),$$

the integral being taken from $x = 0$ to $x = X$. Similarly we have

$$2 \int x \sin n_2 x \, F(x) \, dx = a_2 \left( X - \frac{1}{2n_2} \sin 2n_2 X \right).$$

In the same manner all the following coefficients may be determined. It is easy to see that the definite integral $2 \int x \sin nx \, F(x) \, dx$ always has a determinate value, whatever the arbitrary function $F(x)$ may be. If the function $F(x)$ be represented by the variable ordinate of a line traced in any manner, the function $xF(x) \sin nx$ corresponds to the ordinate of a second line which can easily be constructed by means of the first. The area bounded by the latter line between the abscissæ $x = 0$ and $x = X$ determines the coefficient $a_v$, $v$ being the index of the order of the root $n$.

The arbitrary function $F(x)$ enters each coefficient under the sign of integration, and gives to the value of $v$ all the generality which the problem requires; thus we arrive at the following equation

$$\frac{xy}{2} = \frac{\sin n_1 x \int x \sin n_1 x \, F(x) \, dx}{X - \frac{1}{2n_1} \sin 2n_1 X} e^{-kn_1^2 t} + \frac{\sin n_2 x \int x \sin n_2 x \, F(x) \, dx}{X - \frac{1}{2n_2} \sin 2n_2 X} e^{-kn_2^2 t} + \text{&c.}$$

This is the form which must be given to the general integral of the equation

$$\frac{dv}{dt} = k \left( \frac{d^2 v}{dx^2} + \frac{2}{x} \frac{dv}{dx} \right),$$

in order that it may represent the movement of heat in a solid sphere. In fact, all the conditions of the problem are obeyed.
1st, The partial differential equation is satisfied; 2nd, the quantity of heat which escapes at the surface accords at the same time with the mutual action of the last layers and with the action of the air on the surface; that is to say, the equation \( \frac{dv}{dx} + hx = 0 \), which each part of the value of \( v \) satisfies when \( x = X \), holds also when we take for \( v \) the sum of all these parts; 3rd, the given solution agrees with the initial state when we suppose the time nothing.

292. The roots \( n_1, n_2, n_3, \&c. \) of the equation

\[
\frac{nX}{\tan nX} = 1 - hX
\]

are very unequal; whence we conclude that if the value of the time is considerable, each term of the value of \( v \) is very small, relatively to that which precedes it. As the time of cooling increases, the latter parts of the value of \( v \) cease to have any sensible influence; and those partial and elementary states, which at first compose the general movement, in order that the initial state may be represented by them, disappear almost entirely, one only excepted. In the ultimate state the temperatures of the different layers decrease from the centre to the surface in the same manner as in a circle the ratios of the sine to the arc decrease as the arc increases. This law governs naturally the distribution of heat in a solid sphere. When it begins to exist, it exists through the whole duration of the cooling. Whatever the function \( F(x) \) may be which represents the initial state, the law in question tends continually to be established; and when the cooling has lasted some time, we may without sensible error suppose it to exist.

293. We shall apply the general solution to the case in which the sphere, having been for a long time immersed in a fluid, has acquired at all its points the same temperature. In this case the function \( F(x) \) is 1, and the determination of the coefficients is reduced to integrating \( x \sin nx \, dx \), from \( x = 0 \) to \( x = X \): the integral is

\[
\frac{\sin nX - nX \cos nX}{n^2}.
\]
Hence the value of each coefficient is expressed thus:

\[ a = \frac{2 \sin nX - nX \cos nX}{n nX - \sin nX \cos nX}; \]

the order of the coefficient is determined by that of the root \( n \), the equation which gives the values of \( n \) being

\[ \frac{nX \cos nX}{\sin nX} = 1 - hX. \]

We therefore find

\[ a = \frac{2 hX}{n nX \csc nX - \cos nX}. \]

It is easy now to form the general value which is given by the equation

\[ \frac{\pi x}{2Xh} = \frac{e^{-kn'^2t} \sin n_1 x}{n_1 (n_1 X \csc n_1 X - \cos n_1 X)} + \frac{e^{-kn'^2t} \sin n_2 x}{n_2 (n_2 X \csc n_2 X - \cos n_2 X)} + \&c. \]

Denoting by \( \epsilon_1, \epsilon_2, \epsilon_3, \&c. \) the roots of the equation

\[ \frac{e}{\tan \epsilon} = 1 - hX, \]

and supposing them arranged in order beginning with the least; replacing \( n_1 X, n_2 X, n_3 X, \&c. \) by \( \epsilon_1, \epsilon_2, \epsilon_3, \&c. \), and writing instead of \( k \) and \( h \) their values \( \frac{K}{CD} \) and \( \frac{h}{K} \), we have for the expression of the variations of temperature during the cooling of a solid sphere, which was once uniformly heated, the equation

\[ v = \frac{2h}{K X} \left\{ \frac{\sin \frac{\epsilon_1 x}{X}}{\epsilon_1 \csc \epsilon_1 - \cos \epsilon_1} \right\} - \frac{\epsilon_1^2 \epsilon_2^2}{\epsilon_2 \csc \epsilon_2 - \cos \epsilon_2} + \&c. \]

\( \epsilon \) is a root which satisfies the equation

\[ \tan \epsilon = 1 - \frac{h X}{K}. \]

Note. The problem of the sphere has been very completely discussed by Riemann, *Partielle Differentialgleichungen*, §§ 61—69. [A. F.]
SECTION II.

Different remarks on this solution.

294. We will now explain some of the results which may be derived from the foregoing solution. If we suppose the coefficient \( h \), which measures the facility with which heat passes into the air, to have a very small value, or that the radius \( X \) of the sphere is very small, the least value of \( \varepsilon \) becomes very small; so that the equation

\[
\frac{\varepsilon}{\tan \varepsilon} = 1 - \frac{h}{K} X
\]

is reduced to

\[
\frac{\varepsilon \left(1 - \frac{1}{2} \varepsilon^2\right)}{1 - \frac{3}{2} \varepsilon^2} = 1 - \frac{hX}{K},
\]

or, omitting the higher powers of \( \varepsilon \), \( \varepsilon^2 = \frac{3hX}{K} \). On the other hand, the quantity \( \frac{\varepsilon}{\sin \varepsilon} - \cos \varepsilon \) becomes, on the same hypothesis,

\[
\frac{2hX}{K}.
\]

And the term \( \frac{\sin \frac{\varepsilon v}{X}}{\varepsilon v} \) is reduced to 1. On making these substitutions in the general equation we have \( v = e^{-\frac{3h}{CDX} t} + \&c. \)

We may remark that the succeeding terms decrease very rapidly in comparison with the first, since the second root \( n_2 \) is very much greater than 0; so that if either of the quantities \( h \) or \( X \) has a small value, we may take, as the expression of the variations of temperature, the equation \( v = e^{-\frac{3h}{CDX} t} \). Thus the different spherical envelopes of which the solid is composed retain a common temperature during the whole of the cooling. The temperature diminishes as the ordinate of a logarithmic curve, the time being taken for abscissa; the initial temperature 1 is reduced after the time \( t \) to \( e^{-\frac{3h}{CDX}} \). In order that the initial temperature may be reduced to the fraction \( \frac{1}{m} \), the value of \( t \) must be \( \frac{X}{3h} CD \log m \). Thus in spheres of the same material but
of different diameters, the times occupied in losing half or the same defined part of their actual heat, when the exterior conducibility is very small, are proportional to their diameters. The same is the case with solid spheres whose radius is very small; and we should also find the same result on attributing to the interior conducibility $K$ a very great value. The statement holds generally when the quantity $\frac{hX}{K}$ is very small. We may regard the quantity $\frac{h}{K}$ as very small when the body which is being cooled is formed of a liquid continually agitated, and enclosed in a spherical vessel of small thickness. The hypothesis is in some measure the same as that of perfect conducibility; the temperature decreases then according to the law expressed by the equation $v = e^{-\frac{3ht}{CDX}}$.

295. By the preceding remarks we see that in a solid sphere which has been cooling for a long time, the temperature decreases from the centre to the surface as the quotient of the sine by the arc decreases from the origin where it is 1 to the end of a given arc $e$, the radius of each layer being represented by the variable length of that arc. If the sphere has a small diameter, or if its interior conducibility is very much greater than the exterior conducibility, the temperatures of the successive layers differ very little from each other, since the whole arc $e$ which represents the radius $X$ of the sphere is of small length. The variation of the temperature $v$ common to all its points is then given by the equation $v = e^{-\frac{3ht}{CDX}}$. Thus, on comparing the respective times which two small spheres occupy in losing half or any aliquot part of their actual heat, we find those times to be proportional to the diameters.

296. The result expressed by the equation $v = e^{-\frac{3ht}{CDX}}$ belongs only to masses of similar form and small dimension. It has been known for a long time by physicists, and it offers itself as it were spontaneously. In fact, if any body is sufficiently small for the temperatures at its different points to be regarded as equal, it is easy to ascertain the law of cooling. Let 1 be the initial
temperature common to all points; it is evident that the quantity of heat which flows during the instant \(dt\) into the medium supposed to be maintained at temperature 0 is \(hSvd\), denoting by \(S\) the external surface of the body. On the other hand, if \(C\) is the heat required to raise unit of weight from the temperature 0 to the temperature 1, we shall have \(CDV\) for the expression of the quantity of heat which the volume \(V\) of the body whose density is \(D\) would take from temperature 0 to temperature 1. Hence \(\frac{hSvd}{CDV}\) is the quantity by which the temperature \(v\) is diminished when the body loses a quantity of heat equal to \(hSvd\). We ought therefore to have the equation

\[
\frac{dv}{dt} = -\frac{hSvd}{CDV}, \quad \text{or} \quad v = e^{-\frac{hS}{CDV}t}.
\]

If the form of the body is a sphere whose radius is \(X\), we shall have the equation \(v = e^{-\frac{hS}{CDX}}\).

297. Assuming that we observe during the cooling of the body in question two temperatures \(v_1\) and \(v_2\) corresponding to the times \(t_1\) and \(t_2\), we have

\[
\frac{hS}{CDV} = \frac{\log v_1 - \log v_2}{t_2 - t_1}.
\]

We can then easily ascertain by experiment the exponent \(\frac{hS}{CDV}\).

If the same observation be made on different bodies, and if we know in advance the ratio of their specific heats \(C\) and \(C'\), we can find that of their exterior conducibilities \(h\) and \(h'\). Reciprocally, if we have reason to regard as equal the values \(h\) and \(h'\) of the exterior conducibilities of two different bodies, we can ascertain the ratio of their specific heats. We see by this that, by observing the times of cooling for different liquids and other substances enclosed successively in the same vessel whose thickness is small, we can determine exactly the specific heats of those substances.

We may further remark that the coefficient \(K\) which measures the interior conducibility does not enter into the equation

\[
v = e^{-\frac{hS}{CDX}}.
\]
Thus the time of cooling in bodies of small dimension does not depend on the interior conducibility; and the observation of these times can teach us nothing about the latter property; but it could be determined by measuring the times of cooling in vessels of different thicknesses.

298. What we have said above on the cooling of a sphere of small dimension, applies to the movement of heat in a thermometer surrounded by air or fluid. We shall add the following remarks on the use of these instruments.

Suppose a mercurial thermometer to be dipped into a vessel filled with hot water, and that the vessel is being cooled freely in air at constant temperature. It is required to find the law of the successive falls of temperature of the thermometer.

If the temperature of the fluid were constant, and the thermometer dipped in it, its temperature would change, approaching very quickly that of the fluid. Let \( v \) be the variable temperature indicated by the thermometer, that is to say, its elevation above the temperature of the air; let \( u \) be the elevation of temperature of the fluid above that of the air, and \( t \) the time corresponding to these two values \( v \) and \( u \). At the beginning of the instant \( dt \) which is about to elapse, the difference of the temperature of the thermometer from that of the fluid being \( v - u \), the variable \( v \) tends to diminish and will lose in the instant \( dt \) a quantity proportional to \( v - u \); so that we have the equation

\[
\frac{dv}{dt} = -h(v - u)dt.
\]

During the same instant \( dt \) the variable \( u \) tends to diminish, and it loses a quantity proportional to \( u \), so that we have the equation

\[
\frac{du}{dt} = -Hu dt.
\]

The coefficient \( H \) expresses the velocity of the cooling of the liquid in air, a quantity which may easily be discovered by experiment, and the coefficient \( h \) expresses the velocity with which the thermometer cools in the liquid. The latter velocity is very much greater than \( H \). Similarly we may from experiment find the coefficient \( h \) by making the thermometer cool in fluid maintained at a constant temperature. The two equations

\[
\frac{du}{dt} = -Hu dt \quad \text{and} \quad \frac{dv}{dt} = -h(v - u)dt,
\]
or 
\[ u = Ae^{-Ht} \text{ and } \frac{dv}{dt} = -hv + hAe^{-Ht} \]

lead to the equation
\[ v - u = be^{-ht} + aHe^{-Ht}, \]

\( a \) and \( b \) being arbitrary constants. Suppose now the initial value of \( v - u \) to be \( \Delta \), that is, that the height of the thermometer exceeds by \( \Delta \) the true temperature of the fluid at the beginning of the immersion; and that the initial value of \( u \) is \( E \). We can determine \( a \) and \( b \), and we shall have

\[ v - u = \Delta e^{-ht} + \frac{HE}{h - H}(e^{-Ht} - e^{-ht}). \]

The quantity \( v - u \) is the error of the thermometer, that is to say, the difference which is found between the temperature indicated by the thermometer and the real temperature of the fluid at the same instant. This difference is variable, and the preceding equation informs us according to what law it tends to decrease. We see by the expression for the difference \( v - u \) that two of its terms containing \( e^{-ht} \) diminish very rapidly, with the velocity which would be observed in the thermometer if it were dipped into fluid at constant temperature. With respect to the term which contains \( e^{-Ht} \), its decrease is much slower, and is effected with the velocity of cooling of the vessel in air. It follows from this, that after a time of no great length the error of the thermometer is represented by the single term

\[ \frac{HE}{h - H} e^{-Ht} \text{ or } \frac{H}{h - H} u. \]

299. Consider now what experiment teaches as to the values of \( H \) and \( h \). Into water at 85° (octogesimal scale) we dipped a thermometer which had first been heated, and it descended in the water from 40 to 20 degrees in six seconds. This experiment was repeated carefully several times. From this we find that the value of \( e^{-h} \) is 0.000042; if the time is reckoned in minutes, that is to say, if the height of the thermometer be \( E \) at the beginning of a minute, it will be \( E(0.000042) \) at the end of the minute. Thus we find

\[ h \log_{10} e = 4.3761271. \]

\(^1\) 0.00004206, strictly. [A. F.]
At the same time a vessel of porcelain filled with water heated to 60° was allowed to cool in air at 12°. The value of \( e^{-H} \) in this case was found to be 0.98514, hence that of \( H \log_{10} e \) is 0.006500. We see by this how small the value of the fraction \( e^{-h} \) is, and that after a single minute each term multiplied by \( e^{-ht} \) is not half the ten-thousandth part of what it was at the beginning of the minute. We need not therefore take account of those terms in the value of \( v - u \). The equation becomes

\[
v - u = \frac{Hu}{h - H} \quad \text{or} \quad v - u = \frac{Hu}{h} + \frac{H}{H - h} \frac{Hu}{h}.
\]

From the values found for \( H \) and \( h \), we see that the latter quantity \( h \) is more than 673 times greater than \( H \), that is to say, the thermometer cools in air more than 600 times faster than the vessel cools in air. Thus the term \( \frac{Hu}{h} \) is certainly less than the 600th part of the elevation of temperature of the water above that of the air, and as the term \( \frac{H}{h - H} \frac{Hu}{h} \) is less than the 600th part of the preceding term, which is already very small, it follows that the equation which we may employ to represent very exactly the error of the thermometer is

\[
v - u = \frac{Hu}{h}.
\]

In general if \( H \) is a quantity very great relatively to \( h \), we have always the equation

\[
v - u = \frac{Hu}{h}.
\]

300. The investigation which we have just made furnishes very useful results for the comparison of thermometers.

The temperature marked by a thermometer dipped into a fluid which is cooling is always a little greater than that of the fluid. This excess or error of the thermometer differs with the height of the thermometer. The amount of the correction will be found by multiplying \( u \) the actual height of the thermometer by the ratio of \( H \), the velocity of cooling of the vessel in air, to \( h \) the velocity of cooling of the thermometer in the fluid. We might suppose that the thermometer, when it was dipped into
the fluid, marked a lower temperature. This is what almost always happens, but this state cannot last, the thermometer begins to approach to the temperature of the fluid; at the same time the fluid cools, so that the thermometer passes first to the same temperature as the fluid, and it then indicates a temperature very slightly different but always higher.

300*. We see by these results that if we dip different thermometers into the same vessel filled with fluid which is cooling slowly, they must all indicate very nearly the same temperature at the same instant. Calling \( h \), \( h' \), \( h'' \), the velocities of cooling of the thermometers in the fluid, we shall have

\[
\begin{align*}
\frac{Hu}{h}, \quad \frac{Hu}{h'}, \quad \frac{Hu}{h''}
\end{align*}
\]

as their respective errors. If two thermometers are equally sensitive, that is to say if the quantities \( h \) and \( h' \) are the same, their temperatures will differ equally from those of the fluid. The values of the coefficients \( h, h', h'' \) are very great, so that the errors of the thermometers are extremely small and often inappreciable quantities. We conclude from this that if a thermometer is constructed with care and can be regarded as exact, it will be easy to construct several other thermometers of equal exactness. It will be sufficient to place all the thermometers which we wish to graduate in a vessel filled with a fluid which cools slowly, and to place in it at the same time the thermometer which ought to serve as a model; we shall only have to observe all from degree to degree, or at greater intervals, and we must mark the points where the mercury is found at the same time in the different thermometers. These points will be at the divisions required. We have applied this process to the construction of the thermometers employed in our experiments, so that these instruments coincide always in similar circumstances.

This comparison of thermometers during the time of cooling not only establishes a perfect coincidence among them, and renders them all similar to a single model; but from it we derive also the means of exactly dividing the tube of the principal thermometer, by which all the others ought to be regulated. In this way we
satisfy the fundamental condition of the instrument, which is, that any two intervals on the scale which include the same number of degrees should contain the same quantity of mercury. For the rest we omit here several details which do not directly belong to the object of our work.

301. We have determined in the preceding articles the temperature \( v \) received after the lapse of a time \( t \) by an interior spherical layer at a distance \( x \) from the centre. It is required now to calculate the value of the mean temperature of the sphere, or that which the solid would have if the whole quantity of heat which it contains were equally distributed throughout the whole mass. The volume of a sphere whose radius is \( x \) being \( \frac{4\pi x^3}{3} \), the quantity of heat contained in a spherical envelope whose temperature is \( v \), and radius \( x \), will be \( v d\left(\frac{4\pi x^3}{3}\right) \). Hence the mean temperature is

\[
\int \frac{v \cdot \left(\frac{4\pi x^3}{3}\right)}{\frac{4\pi X^3}{3}} \text{ or } \frac{3}{X^3} \int x^2 v dx,
\]

the integral being taken from \( x = 0 \) to \( x = X \). Substitute for \( v \) its value

\[
\frac{a_1}{x} e^{-k_n X} \sin n_1 x + \frac{a_2}{x} e^{-k_n X} \sin n_2 x + \frac{a_3}{x} e^{-k_n X} \sin n_3 x + \text{etc.}
\]

and we shall have the equation

\[
\frac{3}{X^3} \int x^2 v dx = \frac{3}{X^3} \left\{ \frac{\sin n_1 X - n_1 X \cos n_1 X}{n_1^2} e^{-k_n X} \right. \\
+ \frac{\sin n_2 X - n_2 X \cos n_2 X}{n_2^2} e^{-k_n X} + \text{etc.} \right\}.
\]

We found formerly (Art. 293)

\[
a_i = \frac{2 \sin n_i X - n_i X \cos n_i X}{n_i^2 \left(2n_i X - \frac{1}{2} \sin 2n_i X \right)}.
\]
We have, therefore, if we denote the mean temperature by $z$,

$$z = \frac{(\sin \epsilon_1 - \epsilon_1 \cos \epsilon_1)^2}{3 \cdot 4} e^{-\frac{K\epsilon_1^2}{CDX^2}} + \frac{(\sin \epsilon_2 - \epsilon_2 \cos \epsilon_2)^2}{\epsilon_2^2 (2\epsilon_2 - \sin 2\epsilon_2)} e^{-\frac{Ke_2^2}{CDX^2}} + \&c.,$$

an equation in which the coefficients of the exponentials are all positive.

302. Let us consider the case in which, all other conditions remaining the same, the value $X$ of the radius of the sphere becomes infinitely great. Taking up the construction described in Art. 285, we see that since the quantity $\frac{hX}{K}$ becomes infinite, the straight line drawn through the origin cutting the different branches of the curve coincides with the axis of $x$. We find then for the different values of $\epsilon$ the quantities $\pi$, $2\pi$, $3\pi$, etc.

Since the term in the value of $z$ which contains $e^{-\frac{K\epsilon_1^2}{CDX^2}}$ becomes, as the time increases, very much greater than the following terms, the value of $z$ after a certain time is expressed without sensible error by the first term only. The index $\frac{Kn^2}{CD}$ being equal to $\frac{K\pi^2}{CDX^3}$, we see that the final cooling is very slow in spheres of great diameter, and that the index of $\epsilon$ which measures the velocity of cooling is inversely as the square of the diameter.

303. From the foregoing remarks we can form an exact idea of the variations to which the temperatures are subject during the cooling of a solid sphere. The initial values of the temperatures change successively as the heat is dissipated through the surface. If the temperatures of the different layers are at first equal, or if they diminish from the surface to the centre, they do not maintain their first ratios, and in all cases the system tends more and more towards a lasting state, which after no long delay is sensibly attained. In this final state the temperatures decrease

---

1 Riemann has shewn, Part. Diff. gleich. § 69, that in the case of a very large sphere, uniformly heated initially, the surface temperature varies ultimately as the square root of the time inversely. [A. F.]
from the centre to the surface. If we represent the whole radius of the sphere by a certain arc \( e \) less than a quarter of the circumference, and, after dividing this arc into equal parts, take for each point the quotient of the sine by the arc, this system of ratios will represent that which is of itself set up among the temperatures of layers of equal thickness. From the time when these ultimate ratios occur they continue to exist throughout the whole of the cooling. Each of the temperatures then diminishes as the ordinate of a logarithmic curve, the time being taken for abscissa. We can ascertain that this law is established by observing several successive values \( z, z', z'', z''' \), etc., which denote the mean temperature for the times \( t, t + \Theta, t + 2\Theta, t + 3\Theta \), etc.; the series of these values converges always towards a geometrical progression, and when the successive quotients \( \frac{z'}{z}, \frac{z''}{z'}, \frac{z'''}{z''} \), etc. no longer change, we conclude that the relations in question are established between the temperatures. When the diameter of the sphere is small, these quotients become sensibly equal as soon as the body begins to cool. The duration of the cooling for a given interval, that is to say the time required for the mean temperature \( z \) to be reduced to a definite part of itself \( \frac{z}{m} \), increases as the diameter of the sphere is enlarged.

304. If two spheres of the same material and different dimensions have arrived at the final state in which whilst the temperatures are lowered their ratios are preserved, and if we wish to compare the durations of the same degree of cooling in both, that is to say, the time \( \Theta \) which the mean temperature of the first occupies in being reduced to \( \frac{z}{m} \), and the time \( \Theta' \) in which the temperature \( z' \) of the second becomes \( \frac{z'}{m} \), we must consider three different cases. If the diameter of each sphere is small, the durations \( \Theta \) and \( \Theta' \) are in the same ratio as the diameters. If the diameter of each sphere is very great, the durations \( \Theta \) and \( \Theta' \) are in the ratio of the squares of the diameters; and if the diameters of the spheres are included between these two limits, the ratios of the times will be greater than that of the diameters, and less than that of their squares.
The exact value of the ratio has been already determined. The problem of the movement of heat in a sphere includes that of the terrestrial temperatures. In order to treat of this problem at greater length, we have made it the object of a separate chapter.

305. The use which has been made above of the equation \( e = \tan \epsilon \) is founded on a geometrical construction which is very well adapted to explain the nature of these equations. The construction indeed shows clearly that all the roots are real; at the same time it ascertains their limits, and indicates methods for determining the numerical value of each root. The analytical investigation of equations of this kind would give the same results. First, we might ascertain that the equation \( e - \lambda \tan \epsilon = 0 \), in which \( \lambda \) is a known number less than unity, has no imaginary root of the form \( m + n\sqrt{-1} \). It is sufficient to substitute this quantity for \( \epsilon \); and we see after the transformations that the first member cannot vanish when we give to \( m \) and \( n \) real values, unless \( n \) is nothing. It may be proved moreover that there can be no imaginary root of any form whatever in the equation

\[
e - \lambda \tan \epsilon = 0, \quad \text{or} \quad \frac{e \cos \epsilon - \lambda \sin \epsilon}{\cos \epsilon} = 0.
\]

In fact, 1st, the imaginary roots of the factor \( \frac{1}{\cos \epsilon} = 0 \) do not belong to the equation \( e - \lambda \tan \epsilon = 0 \), since these roots are all of the form \( m + n\sqrt{-1} \); 2nd, the equation \( \sin \epsilon - \frac{e}{\lambda} \cos \epsilon = 0 \) has necessarily all its roots real when \( \lambda \) is less than unity. To prove this proposition we must consider \( \sin \epsilon \) as the product of the infinite number of factors

1. It is \( \Theta : \Theta' = \varepsilon_1^2X^2 : \varepsilon_1^2X^2 \), as may be inferred from the exponent of the first term in the expression for \( z \), Art. 301. [A. F.]
2. The chapter referred to is not in this work. It forms part of the Suite du mémorie sur la théorie du mouvement de la chaleur dans les corps solides. See note, page 10.

The first memoir, entitled Théorie du mouvement de la chaleur dans les corps solides, is that which formed the basis of the Théorie analytique du mouvement de la chaleur published in 1822, but was considerably altered and enlarged in that work now translated. [A. F.]
\[ \epsilon \left(1 - \frac{\epsilon}{\pi^2}\right) \left(1 - \frac{\epsilon^2}{2^2 \pi^2}\right) \left(1 - \frac{\epsilon^2}{3^2 \pi^2}\right) \left(1 - \frac{\epsilon^2}{4^2 \pi^2}\right) \ldots \]

and consider \( \cos \epsilon \) as derived from \( \sin \epsilon \) by differentiation.

Suppose that instead of forming \( \sin \epsilon \) from the product of an infinite number of factors, we employ only the \( m \) first, and denote the product by \( \phi_m(\epsilon) \). To find the corresponding value of \( \cos \epsilon \), we take

\[ \frac{d}{d\epsilon} \phi_m(\epsilon) \text{ or } \phi_m'(\epsilon). \]

This done, we have the equation

\[ \phi_m(\epsilon) - \epsilon \phi_m'(\epsilon) = 0. \]

Now, giving to the number \( m \) its successive values 1, 2, 3, 4, \&c. from 1 to infinity, we ascertain by the ordinary principles of Algebra, the nature of the functions of \( \epsilon \) which correspond to these different values of \( m \). We see that, whatever \( m \) the number of factors may be, the equations in \( \epsilon \) which proceed from them have the distinctive character of equations all of whose roots are real. Hence we conclude rigorously that the equation

\[ \frac{\epsilon}{\tan \epsilon} = \lambda, \]

in which \( \lambda \) is less than unity, cannot have an imaginary root\(^1\).

The same proposition could also be deduced by a different analysis which we shall employ in one of the following chapters.

Moreover the solution we have given is not founded on the property which the equation possesses of having all its roots real. It would not therefore have been necessary to prove this proposition by the principles of algebraical analysis. It is sufficient for the accuracy of the solution that the integral can be made to coincide with any initial state whatever; for it follows rigorously that it must then also represent all the subsequent states.

\(^1\) The proof given by Riemann, Part. Diff. Gleich. § 67, is more simple. The method of proof is in part claimed by Poisson, Bulletin de la Société Philomatique, Paris, 1826, p. 147. [A. F.].