CHAPTER II.

EQUATIONS OF THE MOVEMENT OF HEAT.

SECTION I.

Equation of the varied movement of heat in a ring.

101. We might form the general equations which represent the movement of heat in solid bodies of any form whatever, and apply them to particular cases. But this method would often involve very complicated calculations which may easily be avoided. There are several problems which it is preferable to treat in a special manner by expressing the conditions which are appropriate to them; we proceed to adopt this course and examine separately the problems which have been enunciated in the first section of the introduction; we will limit ourselves at first to forming the differential equations, and shall give the integrals of them in the following chapters.

102. We have already considered the uniform movement of heat in a prismatic bar of small thickness whose extremity is immersed in a constant source of heat. This first case offered no difficulties, since there was no reference except to the permanent state of the temperatures, and the equation which expresses them is easily integrated. The following problem requires a more profound investigation; its object is to determine the variable state of a solid ring whose different points have received initial temperatures entirely arbitrary.

The solid ring or armlet is generated by the revolution of a rectangular section about an axis perpendicular to the plane of
the ring (see figure 3), \( l \) is the perimeter of the section whose area is \( S \), the coefficient \( h \) measures the external conducibility, \( K \) the internal conducibility, \( C \) the specific capacity for heat, \( D \) the density. The line \( oxx'x' \) represents the mean circumference of the armlet, or that line which passes through the centres of figure of all the sections; the distance of a section from the origin \( o \) is measured by the arc whose length is \( x \); \( R \) is the radius of the mean circumference.

It is supposed that on account of the small dimensions and of the form of the section, we may consider the temperature at the different points of the same section to be equal.

103. Imagine that initial arbitrary temperatures have been given to the different sections of the armlet, and that the solid is then exposed to air maintained at the temperature 0, and displaced with a constant velocity; the system of temperatures will continually vary, heat will be propagated within the ring, and dispersed at the surface: it is required to determine what will be the state of the solid at any given instant.

Let \( v \) be the temperature which the section situated at distance \( x \) will have acquired after a lapse of time \( t \); \( v \) is a certain function of \( x \) and \( t \), into which all the initial temperatures also must enter: this is the function which is to be discovered.

104. We will consider the movement of heat in an infinitely small slice, enclosed between a section made at distance \( x \) and another section made at distance \( x + dx \). The state of this slice for the duration of one instant is that of an infinite solid terminated by two parallel planes maintained at unequal temperatures; thus the quantity of heat which flows during this instant \( dt \) across the first section, and passes in this way from the part of the solid which precedes the slice into the slice itself, is measured according to the principles established in the introduction, by the product of four factors, that is to say, the conducibility \( K \), the area of the section \( S \), the ratio \( -\frac{dv}{dx} \), and the duration of the instant; its expression is \( -KS\frac{dv}{dx}dt \). To determine the quantity of heat
which escapes from the same slice across the second section, and passes into the contiguous part of the solid, it is only necessary to change $x$ into $x + dx$ in the preceding expression, or, which is the same thing, to add to this expression its differential taken with respect to $x$; thus the slice receives through one of its faces a quantity of heat equal to $-KS \frac{dv}{dx} dt$, and loses through the opposite face a quantity of heat expressed by

$$-KS \frac{dv}{dx} dt - KS \frac{d^2v}{dx^2} dx dt.$$ 

It acquires therefore by reason of its position a quantity of heat equal to the difference of the two preceding quantities, that is

$$KS \frac{d^2v}{dx^2} dx dt.$$

On the other hand, the same slice, whose external surface is $hdlx$ and whose temperature differs infinitely little from $v$, allows a quantity of heat equivalent to $hlv dx dt$ to escape into the air during the instant $dt$; it follows from this that this infinitely small part of the solid retains in reality a quantity of heat represented by $KS \frac{d^2v}{dx^2} dx dt - hlv dx dt$ which makes its temperature vary. The amount of this change must be examined.

105. The coefficient $C$ expresses how much heat is required to raise unit of weight of the substance in question from temperature 0 up to temperature 1; consequently, multiplying the volume $Sdx$ of the infinitely small slice by the density $D$, to obtain its weight, and by $C$ the specific capacity for heat, we shall have $CDS dx$ as the quantity of heat which would raise the volume of the slice from temperature 0 up to temperature 1. Hence the increase of temperature which results from the addition of a quantity of heat equal to $KS \frac{d^2v}{dx^2} dx dt - hlv dx dt$ will be found by dividing the last quantity by $CDS dx$. Denoting therefore, according to custom, the increase of temperature which takes place during the instant $dt$ by $\frac{dv}{dt} dt$, we shall have the equation
We shall explain in the sequel the use which may be made of this equation to determine the complete solution, and what the difficulty of the problem consists in; we limit ourselves here to a remark concerning the permanent state of the armlet.

106. Suppose that, the plane of the ring being horizontal, sources of heat, each of which exerts a constant action, are placed below different points \( m, n, p, q \) etc.; heat will be propagated in the solid, and that which is dissipated through the surface being incessantly replaced by that which emanates from the sources, the temperature of every section of the solid will approach more and more to a stationary value which varies from one section to another. In order to express by means of equation \((b)\) the law of the latter temperatures, which would exist of themselves if they were once established, we must suppose that the quantity \( v \) does not vary with respect to \( t \); which annuls the term \( \frac{dv}{dt} \). We thus have the equation

\[
\frac{d^2v}{dx^2} = \frac{h}{KS} v, \quad \text{whence} \quad v = M e^{-\frac{h}{KS}} + N e^{+\frac{h}{KS}},
\]

\( M \) and \( N \) being two constants\(^1\).

\(^1\) This equation is the same as the equation for the steady temperature of a finite bar heated at one end (Art. 76), except that \( t \) here denotes the perimeter of a section whose area is \( S \). In the case of the finite bar we can determine two relations between the constants \( M \) and \( N \): for, if \( V \) be the temperature at the source, where \( x = 0, V = M + N \); and if at the end of the bar remote from the source, where \( x = L \) suppose, we make a section at a distance \( dx \) from that end, the flow through this section is, in unit of time, \(-KS\frac{dv}{dx}\), and this is equal to the waste of heat through the periphery and free end of the slice, \( hv (dx + S) \) namely; hence ultimately, \( dx \) vanishing,

\[
hv + K \frac{dv}{dx} = 0, \quad \text{when} \quad x = L,
\]

that is

\[
M e^{-\frac{h}{KS}} + N e^{+\frac{h}{KS}} = \sqrt{\frac{KL}{hS}} \left( M e^{-\frac{h}{KS}} - N e^{+\frac{h}{KS}} \right).
\]

Cf. Verdet, *Conférences de Physique*, p. 37. [A. F.]
107. Suppose a portion of the circumference of the ring, situated between two successive sources of heat, to be divided into equal parts, and denote by $v_1, v_2, v_3, v_4, \&c.$, the temperatures at the points of division whose distances from the origin are $x_1, x_2, x_3, x_4, \&c.$; the relation between $v$ and $x$ will be given by the preceding equation, after that the two constants have been determined by means of the two values of $v$ corresponding to the sources of heat. Denoting by $a$ the quantity $e^{-\frac{k^2}{K^2}}$, and by $\lambda$ the distance $x_2 - x_1$ of two consecutive points of division, we shall have the equations:

$$v_1 = Mx^{x_1} + Na^{-x_1},$$
$$v_2 = Mx^{x_1}x^{x_2} + Na^{-x_1}a^{-x_2},$$
$$v_3 = Mx^{2x_1}a^{-x_1} + Na^{-2x_1}a^{-x_1},$$

whence we derive the following relation:

$$\frac{v_1 + v_3}{v_2} = a^x + a^{-x}.$$  

We should find a similar result for the three points whose temperatures are $v_2, v_3, v_4$, and in general for any three consecutive points. It follows from this that if we observed the temperatures $v_1, v_2, v_3, v_4, \&c.$ of several successive points, all situated between the same two sources $m$ and $n$ and separated by a constant interval $\lambda$, we should perceive that any three consecutive temperatures are always such that the sum of the two extremes divided by the mean gives a constant quotient $a^x + a^{-x}$.

108. If, in the space included between the next two sources of heat $n$ and $p$, the temperatures of other different points separated by the same interval $\lambda$ were observed, it would still be found that for any three consecutive points, the sum of the two extreme temperatures, divided by the mean, gives the same quotient $a^x + a^{-x}$. The value of this quotient depends neither on the position nor on the intensity of the sources of heat.

109. Let $q$ be this constant value, we have the equation:

$$v_3 = qv_2 - v;$$

we see by this that when the circumference is divided into equal parts, the temperatures at the points of division, included between
two consecutive sources of heat, are represented by the terms of a recurring series whose scale of relation is composed of two terms $q$ and $-1$.

Experiments have fully confirmed this result. We have exposed a metallic ring to the permanent and simultaneous action of different sources of heat, and we have observed the stationary temperatures of several points separated by constant intervals; we always found that the temperatures of any three consecutive points, not separated by a source of heat, were connected by the relation in question. Even if the sources of heat be multiplied, and in whatever manner they be disposed, no change can be effected in the numerical value of the quotient $\frac{v_1 + v_3}{v_2}$; it depends only on the dimensions or on the nature of the ring, and not on the manner in which that solid is heated.

110. When we have found, by observation, the value of the constant quotient $q$ or $\frac{v_1 + v_3}{v_2}$, the value of $\alpha^\lambda$ may be derived from it by means of the equation $\alpha^\lambda + \alpha^{-\lambda} = q$. One of the roots is $\alpha^\lambda$, and other root is $\alpha^{-\lambda}$. This quantity being determined, we may derive from it the value of the ratio $\frac{\hbar}{K}$, which is $\frac{S}{l} (\log \alpha)^2$. Denoting $\alpha^\lambda$ by $\omega$, we shall have $\omega^2 - q\omega + 1 = 0$. Thus the ratio of the two conducibilities is found by multiplying $\frac{S}{l}$ by the square of the hyperbolic logarithm of one of the roots of the equation $\omega^2 - q\omega + 1 = 0$, and dividing the product by $\lambda^2$.

SECTION II.

Equation of the varied movement of heat in a solid sphere.

111. A solid homogeneous mass, of the form of a sphere, having been immersed for an infinite time in a medium maintained at a permanent temperature 1, is then exposed to air which is kept at temperature 0, and displaced with constant velocity: it is required to determine the successive states of the body during the whole time of the cooling.
Denote by \( x \) the distance of any point whatever from the centre of the sphere, and by \( v \) the temperature of the same point, after a time \( t \) has elapsed; and suppose, to make the problem more general, that the initial temperature, common to all points situated at the distance \( x \) from the centre, is different for different values of \( x \); which is what would have been the case if the immersion had not lasted for an infinite time.

Points of the solid, equally distant from the centre, will not cease to have a common temperature; \( v \) is thus a function of \( x \) and \( t \). When we suppose \( t = 0 \), it is essential that the value of this function should agree with the initial state which is given, and which is entirely arbitrary.

112. We shall consider the instantaneous movement of heat in an infinitely thin shell, bounded by two spherical surfaces whose radii are \( x \) and \( x + dx \): the quantity of heat which, during an infinitely small instant \( dt \), crosses the lesser surface whose radius is \( x \), and so passes from that part of the solid which is nearest to the centre into the spherical shell, is equal to the product of four factors which are the conducibility \( K \), the duration \( dt \), the extent \( 4\pi x^2 \) of surface, and the ratio \( \frac{dv}{dx} \), taken with the negative sign; it is expressed by \(-4K\pi x^2 \frac{dv}{dx} dt\).

To determine the quantity of heat which flows during the same instant through the second surface of the same shell, and passes from this shell into the part of the solid which envelops it, \( x \) must be changed into \( x + dx \), in the preceding expression: that is to say, to the term \(-4K\pi x^2 \frac{dv}{dx} dt\) must be added the differential of this term taken with respect to \( x \). We thus find

\[-4K\pi x^2 \frac{dv}{dx} dt - 4K\pi dx \left( x^2 \frac{dv}{dx} \right) \cdot dt\]

as the expression of the quantity of heat which leaves the spherical shell across its second surface; and if we subtract this quantity from that which enters through the first surface, we shall have

\[4K\pi dx \left( x^2 \frac{dv}{dx} \right) dt\]. This difference is evidently the quantity of
heat which accumulates in the intervening shell, and whose effect is to vary its temperature.

113. The coefficient $C$ denotes the quantity of heat which is necessary to raise, from temperature 0 to temperature 1, a definite unit of weight; $D$ is the weight of unit of volume, $4\pi x^2 dx$ is the volume of the intervening layer, differing from it only by a quantity which may be omitted: hence $4\pi CDx^2 dx$ is the quantity of heat necessary to raise the intervening shell from temperature 0 to temperature 1. Hence it is requisite to divide the quantity of heat which accumulates in this shell by $4\pi CDx^2 dx$, and we shall then find the increase of its temperature $v$ during the time $dt$. We thus obtain the equation

$$dv = \frac{K}{CD} dt \cdot \frac{d \left( x^2 \frac{dv}{dx} \right)}{x^2 dx},$$

or

$$\frac{dv}{dt} = \frac{K}{CD} \cdot \left( \frac{d^2 v}{dx^2} + \frac{2}{x} \frac{dv}{dx} \right) \ldots \ldots \ldots (c).$$

114. The preceding equation represents the law of the movement of heat in the interior of the solid, but the temperatures of points in the surface are subject also to a special condition which must be expressed. This condition relative to the state of the surface may vary according to the nature of the problems discussed: we may suppose for example, that, after having heated the sphere, and raised all its molecules to the temperature of boiling water, the cooling is effected by giving to all points in the surface the temperature 0, and by retaining them at this temperature by any external cause whatever. In this case we may imagine the sphere, whose variable state it is desired to determine, to be covered by a very thin envelope on which the cooling agency exerts its action. It may be supposed, 1°, that this infinitely thin envelope adheres to the solid, that it is of the same substance as the solid and that it forms a part of it, like the other portions of the mass; 2°, that all the molecules of the envelope are subjected to temperature 0 by a cause always in action which prevents the temperature from ever being above or below zero. To express this condition theoretically, the function $v$, which contains $x$ and $t$,
must be made to become nul, when we give to $x$ its complete value $X$ equal to the radius of the sphere, whatever else the value of $t$ may be. We should then have, on this hypothesis, if we denote by $\phi (x, t)$ the function of $x$ and $t$, which expresses the value of $v$, the two equations

$$\frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dx^2} + \frac{2}{x} \frac{dv}{dx} \right), \quad \text{and} \quad \phi (X, t) = 0.$$  

Further, it is necessary that the initial state should be represented by the same function $\phi (x, t)$: we shall therefore have as a second condition $\phi (x, 0) = 1$. Thus the variable state of a solid sphere on the hypothesis which we have first described will be represented by a function $v$, which must satisfy the three preceding equations. The first is general, and belongs at every instant to all points of the mass; the second affects only the molecules at the surface, and the third belongs only to the initial state.

115. If the solid is being cooled in air, the second equation is different; it must then be imagined that the very thin envelope is maintained by some external cause, in a state such as to produce the escape from the sphere, at every instant, of a quantity of heat equal to that which the presence of the medium can carry away from it.

Now the quantity of heat which, during an infinitely small instant $dt$, flows within the interior of the solid across the spherical surface situate at distance $x$, is equal to $-4K\pi x^2 \frac{dv}{dx} dt$; and this general expression is applicable to all values of $x$. Thus, by supposing $x = X$ we shall ascertain the quantity of heat which in the variable state of the sphere would pass across the very thin envelope which bounds it; on the other hand, the external surface of the solid having a variable temperature, which we shall denote by $V$, would permit the escape into the air of a quantity of heat proportional to that temperature, and to the extent of the surface, which is $4\pi X^2$. The value of this quantity is $4h\pi X^2 V dt$.

To express, as is supposed, that the action of the envelope supplies the place, at every instant, of that which would result from the presence of the medium, it is sufficient to equate the quantity $4h\pi X^2 V dt$ to the value which the expression $-4K\pi X^2 \frac{dv}{dx} dt$
receives when we give to $x$ its complete value $X$; hence we obtain the equation $\frac{dv}{dx} = -\frac{h}{K} v$, which must hold when in the functions $\frac{dv}{dx}$ and $v$ we put instead of $x$ its value $X$, which we shall denote by writing it in the form $K \frac{dV}{dx} + hV = 0$.

116. The value of $\frac{dv}{dx}$ taken when $x = X$, must therefore have a constant ratio $-\frac{h}{K}$ to the value of $v$, which corresponds to the same point. Thus we shall suppose that the external cause of the cooling determines always the state of the very thin envelope, in such a manner that the value of $\frac{dv}{dx}$ which results from this state, is proportional to the value of $v$, corresponding to $x = X$, and that the constant ratio of these two quantities is $-\frac{h}{K}$. This condition being fulfilled by means of some cause always present, which prevents the extreme value of $\frac{dv}{dx}$ from being anything else but $-\frac{h}{K} v$, the action of the envelope will take the place of that of the air.

It is not necessary to suppose the envelope to be extremely thin, and it will be seen in the sequel that it may have an indefinite thickness. Here the thickness is considered to be indefinitely small, so as to fix the attention on the state of the surface only of the solid.

117. Hence it follows that the three equations which are required to determine the function $\phi(x, t)$ or $v$ are the following,

$$ \frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2 v}{dx^2} + \frac{2}{x} \frac{dv}{dx} \right), \quad K \frac{dV}{dx} + hV = 0, \quad \phi(x, 0) = 1. $$

The first applies to all possible values of $x$ and $t$; the second is satisfied when $x = X$, whatever be the value of $t$; and the third is satisfied when $t = 0$, whatever be the value of $x$. 
It might be supposed that in the initial state all the spherical layers have not the same temperature: which is what would necessarily happen, if the immersion were imagined not to have lasted for an indefinite time. In this case, which is more general than the foregoing, the given function, which expresses the initial temperature of the molecules situated at distance $x$ from the centre of the sphere, will be represented by $F(x)$; the third equation will then be replaced by the following, $\phi(x, 0) = F(x)$.

Nothing more remains than a purely analytical problem, whose solution will be given in one of the following chapters. It consists in finding the value of $v$, by means of the general condition, and the two special conditions to which it is subject.

**SECTION III.**

**Equations of the varied movement of heat in a solid cylinder.**

118. A solid cylinder of infinite length, whose side is perpendicular to its circular base, having been wholly immersed in a liquid whose temperature is uniform, has been gradually heated, in such a manner that all points equally distant from the axis have acquired the same temperature; it is then exposed to a current of colder air; it is required to determine the temperatures of the different layers, after a given time.

$x$ denotes the radius of a cylindrical surface, all of whose points are equally distant from the axis; $X$ is the radius of the cylinder; $v$ is the temperature which points of the solid, situated at distance $x$ from the axis, must have after the lapse of a time denoted by $t$, since the beginning of the cooling. Thus $v$ is a function of $x$ and $t$, and if in it $t$ be made equal to 0, the function of $x$ which arises from this must necessarily satisfy the initial state, which is arbitrary.

119. Consider the movement of heat in an infinitely thin portion of the cylinder, included between the surface whose radius is $x$, and that whose radius is $x + dx$. The quantity of heat which this portion receives during the instant $dt$, from the part of the solid which it envelops, that is to say, the quantity which during the same time crosses the cylindrical surface
whose radius is \( x \), and whose length is supposed to be equal to unity, is expressed by

\[
-2K\pi x \frac{dv}{dx} dt.
\]

To find the quantity of heat which, crossing the second surface whose radius is \( x + dx \), passes from the infinitely thin shell into the part of the solid which envelops it, we must, in the foregoing expression, change \( x \) into \( x + dx \), or, which is the same thing, add to the term

\[
-2K\pi x \frac{dv}{dx} dt,
\]

the differential of this term, taken with respect to \( x \). Hence the difference of the heat received and the heat lost, or the quantity of heat which accumulating in the infinitely thin shell determines the changes of temperature, is the same differential taken with the opposite sign, or

\[
2K\pi . dt . d \left( x \frac{dv}{dx} \right);
\]
on the other hand, the volume of this intervening shell is \( 2\pi x dx \), and \( 2CD\pi x dx \) expresses the quantity of heat required to raise it from the temperature 0 to the temperature 1, \( C \) being the specific heat, and \( D \) the density. Hence the quotient

\[
\frac{2K\pi . dt . d \left( x \frac{dv}{dx} \right)}{2CD\pi x dx}
\]
is the increment which the temperature receives during the instant \( dt \). Whence we obtain the equation

\[
\frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dx^2} + \frac{1}{x} \frac{dv}{dx} \right).
\]

120. The quantity of heat which, during the instant \( dt \), crosses the cylindrical surface whose radius is \( x \), being expressed in general by \( 2K\pi x \frac{dv}{dx} dt \), we shall find that quantity which escapes during the same time from the surface of the solid, by making \( x = X \) in the foregoing value; on the other hand, the
same quantity, dispersed into the air, is, by the principle of the communication of heat, equal to $2\pi X h v_0 dt$; we must therefore have at the surface the definite equation $-K \frac{dv}{dx} = hv$. The nature of these equations is explained at greater length, either in the articles which refer to the sphere, or in those wherein the general equations have been given for a body of any form whatever. The function $v$ which represents the movement of heat in an infinite cylinder must therefore satisfy, 1st, the general equation

$$\frac{dv}{dt} = K \left( \frac{d^2v}{dx^2} + \frac{1}{x} \frac{dv}{dx} \right),$$

which applies whatever $x$ and $t$ may be; 2nd, the definite equation $\frac{h}{K} v + \frac{dv}{dx} = 0$, which is true, whatever the variable $t$ may be, when $x = X$; 3rd, the definite equation $v = F(x)$. The last condition must be satisfied by all values of $v$, when $t$ is made equal to 0, whatever the variable $x$ may be. The arbitrary function $F(x)$ is supposed to be known; it corresponds to the initial state.

SECTION IV.

Equations of the uniform movement of heat in a solid prism of infinite length.

121. A prismatic bar is immersed at one extremity in a constant source of heat which maintains that extremity at the temperature $A$; the rest of the bar, whose length is infinite, continues to be exposed to a uniform current of atmospheric air maintained at temperature 0; it is required to determine the highest temperature which a given point of the bar can acquire.

The problem differs from that of Article 73, since we now take into consideration all the dimensions of the solid, which is necessary in order to obtain an exact solution.

We are led, indeed, to suppose that in a bar of very small thickness all points of the same section would acquire sensibly equal temperatures; but some uncertainty may rest on the results of this hypothesis. It is therefore preferable to solve the problem rigorously, and then to examine, by analysis, up to what point, and in what cases, we are justified in considering the temperatures of different points of the same section to be equal.
122. The section made at right angles to the length of the bar, is a square whose side is $2l$, the axis of the bar is the axis of $x$, and the origin is at the extremity $A$. The three rectangular co-ordinates of a point of the bar are $x, y, z$, and $v$ denotes the fixed temperature at the same point.

The problem consists in determining the temperatures which must be assigned to different points of the bar, in order that they may continue to exist without any change, so long as the extreme surface $A$, which communicates with the source of heat, remains subject, at all its points, to the permanent temperature $A$; thus $v$ is a function of $x, y, z$.

123. Consider the movement of heat in a prismatic molecule, enclosed between six planes perpendicular to the three axes of $x, y, z$. The first three planes pass through the point $m$ whose co-ordinates are $x, y, z$, and the others pass through the point $m'$ whose co-ordinates are $x + dx, y + dy, z + dz$.

To find what quantity of heat enters the molecule during unit of time across the first plane passing through the point $m$ and perpendicular to $x$, we must remember that the extent of the surface of the molecule on this plane is $dydz$, and that the flow across this area is, according to the theorem of Article 98, equal to $-K \frac{dv}{dx}$; thus the molecule receives across the rectangle $dydz$ passing through the point $m$ a quantity of heat expressed by $-K dydz \frac{dv}{dx}$. To find the quantity of heat which crosses the opposite face, and escapes from the molecule, we must substitute, in the preceding expression, $x + dx$ for $x$, or, which is the same thing, add to this expression its differential taken with respect to $x$ only; whence we conclude that the molecule loses, at its second face perpendicular to $x$, a quantity of heat equal to

$$-K dydz \frac{dv}{dx} - K dydz d\left(\frac{dv}{dx}\right);$$

we must therefore subtract this from that which enters at the opposite face; the differences of these two quantities is

$$K dydz d\left(\frac{dv}{dx}\right), \text{ or, } K dxdydz \frac{dv}{dx};$$
this expresses the quantity of heat accumulated in the molecule in consequence of the propagation in direction of \( x \); which accumulated heat would make the temperature of the molecule vary, if it were not balanced by that which is lost in some other direction.

It is found in the same manner that a quantity of heat equal to \(-Kdzdx \frac{dv}{dy}\) enters the molecule across the plane passing through the point \( m \) perpendicular to \( y \), and that the quantity which escapes at the opposite face is

\[
-Kdzdx \frac{dv}{dy} - Kdzdx \, d(\frac{dv}{dy}),
\]

the last differential being taken with respect to \( y \) only. Hence the difference of the two quantities, or \( Kdxdydz \frac{d^2v}{dy^2} \), expresses the quantity of heat which the molecule acquires, in consequence of the propagation in direction of \( y \).

Lastly, it is proved in the same manner that the molecule acquires, in consequence of the propagation in direction of \( z \), a quantity of heat equal to \( Kdxdydz \frac{d^2v}{dz^2} \). Now, in order that there may be no change of temperature, it is necessary for the molecule to retain as much heat as it contained at first, so that the heat it acquires in one direction must balance that which it loses in another. Hence the sum of the three quantities of heat acquired must be nothing; thus we form the equation

\[
\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0.
\]

124. It remains now to express the conditions relative to the surface. If we suppose the point \( m \) to belong to one of the faces of the prismatic bar, and the face to be perpendicular to \( z \), we see that the rectangle \( dx dy \), during unit of time, permits a quantity of heat equal to \( Vh \, dx \, dy \) to escape into the air, \( V \) denoting the temperature of the point \( m \) of the surface, namely what \( \phi (x, y, z) \) the function sought becomes when \( z \) is made equal to \( l \), half the dimension of the prism. On the other hand, the quantity of heat which, by virtue of the action of the
molecules, during unit of time, traverses an infinitely small surface \( \omega \), situated within the prism, perpendicular to \( z \), is equal to 

\[-K \frac{dv}{dz} \]

according to the theorems quoted above. This expression is general, and applying it to points for which the coordinate \( z \) has its complete value \( l \), we conclude from it that the quantity of heat which traverses the rectangle \( dx \, dy \) taken at the surface is 

\[-K \frac{dv}{dz} \]

giving to \( z \) in the function \( \frac{dv}{dz} \) its complete value \( l \). Hence the two quantities 

\[-K dx \, dy \frac{dv}{dz} \]

and 

\[h \, dx \, dy \, v\]

must be equal, in order that the action of the molecules may agree with that of the medium. This equality must also exist when we give to \( z \) in the functions \( \frac{dv}{dz} \) and \( v \) the value \(-l\), which it has at the face opposite to that first considered. Further, the quantity of heat which crosses an infinitely small surface \( \omega \), perpendicular to the axis of \( y \), being 

\[-K \frac{dv}{dy} \]

it follows that that which flows across a rectangle \( dz \, dx \) taken on a face of the prism perpendicular to \( y \) is 

\[-K \frac{dv}{dy} \]

giving to \( y \) in the function \( \frac{dv}{dy} \) its complete value \( l \). Now this rectangle \( dz \, dx \) permits a quantity of heat expressed by \( hv \, dx \, dy \) to escape into the air; the equation 

\[hv = -K \frac{dv}{dy} \]

becomes therefore necessary, when \( y \) is made equal to \( l \) or \(-l\) in the functions \( v \) and \( \frac{dv}{dy} \).

125. The value of the function \( v \) must by hypothesis be equal to \( A \), when we suppose \( x = 0 \), whatever be the values of \( y \) and \( z \). Thus the required function \( v \) is determined by the following conditions: 1st, for all values of \( x, y, z \), it satisfies the general equation

\[
\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0;
\]

2nd, it satisfies the equation 

\[\frac{h}{K} v + \frac{dv}{dy} = 0, \]

when \( y \) is equal to
SECTION V.

Equations of the varied movement of heat in a solid cube.

126. A solid in the form of a cube, all of whose points have acquired the same temperature, is placed in a uniform current of atmospheric air, maintained at temperature 0. It is required to determine the successive states of the body during the whole time of the cooling.

The centre of the cube is taken as the origin of rectangular coordinates; the three perpendiculars dropped from this point on the faces, are the axes of $x$, $y$, and $z$; $2l$ is the side of the cube, $v$ is the temperature to which a point whose coordinates are $x$, $y$, $z$, is lowered after the time $t$ has elapsed since the commencement of the cooling; the problem consists in determining the function $v$, which depends on $x$, $y$, $z$ and $t$.

127. To form the general equation which $v$ must satisfy, we must ascertain what change of temperature an infinitely small portion of the solid must experience during the instant $dt$, by virtue of the action of the molecules which are extremely near to it. We consider then a prismatic molecule enclosed between six planes at right angles; the first three pass through the point $m$, whose co-ordinates are $x$, $y$, $z$, and the three others, through the point $m'$, whose co-ordinates are $x + dx$, $y + dy$, $z + dz$.

The quantity of heat which during the instant $dt$ passes into the molecule across the first rectangle $dydz$ perpendicular to $x$, is $-Kdydz\frac{dv}{dx}dt$, and that which escapes in the same time from the molecule, through the opposite face, is found by writing $x + dx$ in place of $x$ in the preceding expression, it is

$$-Kdydz\left(\frac{dv}{dx}\right)dt - Kdydzd\left(\frac{dv}{dx}\right)dt.$$
the differential being taken with respect to $x$ only. The quantity of heat which during the instant $dt$ enters the molecule, across the first rectangle $dz\,dx$ perpendicular to the axis of $y$, is

$$-K\,dz\,dx\,\frac{dv}{dy}\,dt,$$

and that which escapes from the molecule during the same instant, by the opposite face, is

$$-K\,dz\,dx\,\frac{dv}{dy}\,dt - K\,dz\,dx\,d\left(\frac{dv}{dy}\right)\,dt,$$

the differential being taken with respect to $y$ only. The quantity of heat which the molecule receives during the instant $dt$, through its lower face, perpendicular to the axis of $z$, is

$$-K\,dx\,dy\,\frac{dv}{dz}\,dt$$

and that which it loses through the opposite face is

$$-K\,dx\,dy\,\frac{dv}{dz}\,dt - K\,dx\,dy\,d\left(\frac{dv}{dz}\right)\,dt,$$

the differential being taken with respect to $z$ only.

The sum of all the quantities of heat which escape from the molecule must now be deducted from the sum of the quantities which it receives, and the difference is that which determines its increase of temperature during the instant: this difference is

$$K\,dy\,dz\,d\left(\frac{dv}{dx}\right)\,dt + K\,dz\,dx\,d\left(\frac{dv}{dy}\right)\,dt + K\,dx\,dy\,d\left(\frac{dv}{dz}\right)\,dt,$$

or $K\,dx\,dy\,dz\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)\,dt$.

128. If the quantity which has just been found be divided by that which is necessary to raise the molecule from the temperature 0 to the temperature 1, the increase of temperature which is effected during the instant $dt$ will become known. Now, the latter quantity is $CD\,dx\,dy\,dz$: for $C$ denotes the capacity of the substance for heat; $D$ its density, and $dx\,dy\,dz$ the volume of the molecule. The movement of heat in the interior of the solid is therefore expressed by the equation

$$\frac{dv}{dt} = \frac{K}{CD} \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) \ldots \ldots (d).$$
129. It remains to form the equations which relate to the state of the surface, which presents no difficulty, in accordance with the principles which we have established. In fact, the quantity of heat which, during the instant \(dt\), crosses the rectangle \(dz\,dy\), traced on a plane perpendicular to \(x\), is \(-K\,dy\,dz\,d\frac{v}{dx}\,dt\).

This result, which applies to all points of the solid, ought to hold when the value of \(x\) is equal to \(l\), half the thickness of the prism. In this case, the rectangle \(dy\,dz\) being situated at the surface, the quantity of heat which crosses it, and is dispersed into the air during the instant \(dt\), is expressed by \(hv\,dy\,dz\,dt\), we ought therefore to have, when \(x = l\), the equation \(hv = -K\,d\frac{v}{dx}\). This condition must also be satisfied when \(x = -l\).

It will be found also that, the quantity of heat which crosses the rectangle \(dz\,dx\) situated on a plane perpendicular to the axis of \(y\) being in general \(-K\,dz\,dx\,d\frac{v}{dz}\), and that which escapes at the surface into the air across the same rectangle being \(hv\,dz\,dx\,dt\), we must have the equation \(hv + K\,d\frac{v}{dy} = 0\), when \(y = l\) or \(-l\).

Lastly, we obtain in like manner the definite equation

\[kv + K\,d\frac{v}{dz} = 0,\]

which is satisfied when \(z = l\) or \(-l\).

130. The function sought, which expresses the varied movement of heat in the interior of a solid of cubic form, must therefore be determined by the following conditions:

1st. It satisfies the general equation

\[d\frac{v}{dt} = \frac{K}{C\,D}\left(d^2v + d^2v + d^2v\right);\]

2nd. It satisfies the three definite equations

\[hv + K\,d\frac{v}{dx} = 0, \quad hv + K\,d\frac{v}{dy} = 0, \quad hv + K\,d\frac{v}{dz} = 0,\]

which hold when \(x = \pm l\), \(y = \pm l\), \(z = \pm l\);
3rd. If in the function $v$ which contains $x, y, z, t$, we make $t = 0$, whatever be the values of $x, y, z$, we ought to have, according to hypothesis, $v = A$, which is the initial and common value of the temperature.

131. The equation arrived at in the preceding problem represents the movement of heat in the interior of all solids. Whatever, in fact, the form of the body may be, it is evident that, by decomposing it into prismatic molecules, we shall obtain this result. We may therefore limit ourselves to demonstrating in this manner the equation of the propagation of heat. But in order to make the exhibition of principles more complete, and that we may collect into a small number of consecutive articles the theorems which serve to establish the general equation of the propagation of heat in the interior of solids, and the equations which relate to the state of the surface, we shall proceed, in the two following sections, to the investigation of these equations, independently of any particular problem, and without reverting to the elementary propositions which we have explained in the introduction.

**SECTION VI.**

**General equation of the propagation of heat in the interior of solids.**

132. **Theorem I.** If the different points of a homogeneous solid mass, enclosed between six planes at right angles, have actual temperatures determined by the linear equation

$$v = A - ax - by - cz, \ldots \ldots (a),$$

and if the molecules situated at the external surface on the six planes which bound the prism are maintained, by any cause whatever, at the temperature expressed by the equation (a): all the molecules situated in the interior of the mass will of themselves retain their actual temperatures, so that there will be no change in the state of the prism.

$v$ denotes the actual temperature of the point whose coordinates are $x, y, z$; $A, a, b, c$, are constant coefficients.

To prove this proposition, consider in the solid any three points whatever $M\mu$, situated on the same straight line $m\mu$, ...
which the point \( M \) divides into two equal parts; denote by \( x, y, z \) the co-ordinates of the point \( M \), and its temperature by \( v \), the co-ordinates of the point \( \mu \) by \( x + \alpha, y + \beta, z + \gamma \), and its temperature by \( w \), the co-ordinates of the point \( m \) by \( x - \alpha, y - \beta, z - \gamma \), and its temperature by \( u \), we shall have

\[
\begin{align*}
v &= A - ax - by - cz, \\
w &= A - a(x + \alpha) - b(y + \beta) - c(z + \gamma), \\
u &= A - a(x - \alpha) - b(y - \beta) - c(z - \gamma),
\end{align*}
\]

whence we conclude that,

\[
v - w = ax + b\beta + c\gamma, \quad \text{and} \quad u - v = ax + b\beta + c\gamma;
\]

therefore

\[
v - w = u - v.
\]

Now the quantity of heat which one point receives from another depends on the distance between the two points and on the difference of their temperatures. Hence the action of the point \( M \) on the point \( \mu \) is equal to the action of \( m \) on \( M \); thus the point \( M \) receives as much heat from \( m \) as it gives up to the point \( \mu \).

We obtain the same result, whatever be the direction and magnitude of the line which passes through the point \( M \), and is divided into two equal parts. Hence it is impossible for this point to change its temperature, for it receives from all parts as much heat as it gives up.

The same reasoning applies to all other points; hence no change can happen in the state of the solid.

133. Corollary I. A solid being enclosed between two infinite parallel planes \( A \) and \( B \), if the actual temperature of its different points is supposed to be expressed by the equation \( v = 1 - z \), and the two planes which bound it are maintained by any cause whatever, \( A \) at the temperature 1, and \( B \) at the temperature 0; this particular case will then be included in the preceding lemma, if we make \( A = 1, a = 0, b = 0, c = 1 \).

134. Corollary II. If in the interior of the same solid we imagine a plane \( M \) parallel to those which bound it, we see that a certain quantity of heat flows across this plane during unit of time; for two very near points, such as \( m \) and \( n \), one
of which is below the plane and the other above it, are unequally heated; the first, whose temperature is highest, must therefore send to the second, during each instant, a certain quantity of heat which, in some cases, may be very small, and even insensible, according to the nature of the body and the distance of the two molecules.

The same is true for any two other points whatever separated by the plane. That which is most heated sends to the other a certain quantity of heat, and the sum of these partial actions, or of all the quantities of heat sent across the plane, composes a continual flow whose value does not change, since all the molecules preserve their temperatures. It is easy to prove that this flow, or the quantity of heat which crosses the plane $M$ during the unit of time, is equivalent to that which crosses, during the same time, another plane $N$ parallel to the first. In fact, the part of the mass which is enclosed between the two surfaces $M$ and $N$ will receive continually, across the plane $M$, as much heat as it loses across the plane $N$. If the quantity of heat, which in passing the plane $M$ enters the part of the mass which is considered, were not equal to that which escapes by the opposite surface $N$, the solid enclosed between the two surfaces would acquire fresh heat, or would lose a part of that which it has, and its temperatures would not be constant; which is contrary to the preceding lemma.

135. The measure of the specific conducibility of a given substance is taken to be the quantity of heat which, in an infinite solid, formed of this substance, and enclosed between two parallel planes, flows during unit of time across unit of surface, taken on any intermediate plane whatever, parallel to the external planes, the distance between which is equal to unit of length, one of them being maintained at temperature 1, and the other at temperature 0. This constant flow of the heat which crosses the whole extent of the prism is denoted by the coefficient $K$, and is the measure of the conducibility.

136. **Lemma.** If we suppose all the temperatures of the solid in question under the preceding article, to be multiplied by any number whatever $g$, so that the equation of temperatures is $v = g - gz$, instead of being $v = 1 - z$, and if the two external planes are main-
tained, one at the temperature \( g \), and the other at temperature \( 0 \), the constant flow of heat, in this second hypothesis, or the quantity which during unit of time crosses unit of surface taken on an intermediate plane parallel to the bases, is equal to the product of the first flow multiplied by \( g \).

In fact, since all the temperatures have been increased in the ratio of 1 to \( g \), the differences of the temperatures of any two points whatever \( m \) and \( \mu \), are increased in the same ratio. Hence, according to the principle of the communication of heat, in order to ascertain the quantity of heat which \( m \) sends to \( \mu \) on the second hypothesis, we must multiply by \( g \) the quantity which the same point \( m \) sends to \( \mu \) on the first hypothesis. The same would be true for any two other points whatever. Now, the quantity of heat which crosses a plane \( M \) results from the sum of all the actions which the points \( m, m', m'', m''' \), etc., situated on the same side of the plane, exert on the points \( \mu, \mu', \mu'', \mu''' \), etc., situated on the other side. Hence, if in the first hypothesis the constant flow is denoted by \( K \), it will be equal to \( gK \), when we have multiplied all the temperatures by \( g \).

137. **Theorem II.** In a prism whose constant temperatures are expressed by the equation 
\[
v = Ax - by - cz,
\]
and which is bounded by six planes at right angles all of whose points are maintained at constant temperatures determined by the preceding equation, the quantity of heat which, during unit of time, crosses unit of surface taken on any intermediate plane whatever perpendicular to \( z \), is the same as the constant flow in a solid of the same substance would be, if enclosed between two infinite parallel planes, and for which the equation of constant temperatures is 
\[
v = c - cz.
\]

To prove this, let us consider in the prism, and also in the infinite solid, two extremely near points \( m \) and \( \mu \), separated by the plane \( M \) perpendicular to the axis of \( z \); \( \mu \) being above the plane, and \( m \) below it (see fig. 4), and above the same plane...
let us take a point $m$ such that the perpendicular dropped from the point $\mu$ on the plane may also be perpendicular to the distance $mm'$ at its middle point $h$. Denote by $x, y, z + h$, the co-ordinates of the point $\mu$, whose temperature is $w$, by $x - \alpha, y - \beta, z$, the co-ordinates of $m$, whose temperature is $v$, and by $x + \alpha, y + \beta, z$, the co-ordinates of $m'$, whose temperature is $v'$.

The action of $m$ on $\mu$, or the quantity of heat which $m$ sends to $\mu$ during a certain time, may be expressed by $q(v - w)$. The factor $q$ depends on the distance $m\mu$, and on the nature of the mass. The action of $m'$ on $\mu$ will therefore be expressed by $q(v' - w)$; and the factor $q$ is the same as in the preceding expression; hence the sum of the two actions of $m$ on $\mu$, and of $m'$ on $\mu$, or the quantity of heat which $\mu$ receives from $m$ and from $m'$, is expressed by

$$q(v - w + v' - w).$$

Now, if the points $m, \mu, m'$ belong to the prism, we have

$$w = A - ax - by - c(z + h), \quad v = A - a(x - \alpha) - b(y - \beta) - cz,$$

and

$$v' = A - a(x + \alpha) - b(y + \beta) - cz;$$

and if the same points belonged to an infinite solid, we should have, by hypothesis,

$$w = c - c(z + h), \quad v = c - cz, \quad \text{and} \quad v' = c - cz.$$

In the first case, we find

$$q(v - w + v' - w) = 2qch,$$

and, in the second case, we still have the same result. Hence the quantity of heat which $\mu$ receives from $m$ and from $m'$ on the first hypothesis, when the equation of constant temperatures is $v = A - ax - by - cz$, is equivalent to the quantity of heat which $\mu$ receives from $m$ and from $m'$ when the equation of constant temperatures is $v = c - cz$.

The same conclusion might be drawn with respect to any three other points whatever $m', \mu', m''$, provided that the second $\mu'$ be placed at equal distances from the other two, and the altitude of the isosceles triangle $m' \mu' m''$ be parallel to $z$. Now, the quantity of heat which crosses any plane whatever $M$, results from the sum of the actions which all the points $m, m', m'', m'''$ etc., situated on
one side of this plane, exert on all the points \( \mu, \mu', \mu'', \mu''', \) etc situated on the other side: hence the constant flow, which, during unit of time, crosses a definite part of the plane \( M \) in the infinite solid, is equal to the quantity of heat which flows in the same time across the same portion of the plane \( M \) in the prism, all of whose temperatures are expressed by the equation

\[
v = A - ax - by - cz.
\]

138. Corollary. The flow has the value \( cK \) in the infinite solid, when the part of the plane which it crosses has unit of surface. In the prism also it has the same value \( cK \) or \(-K \frac{dv}{dz}\).

It is proved in the same manner, that the constant flow which takes place, during unit of time, in the same prism across unit of surface, on any plane whatever perpendicular to \( y \), is equal to

\[
bK \text{ or } -K \frac{dv}{dy};
\]

and that which crosses a plane perpendicular to \( x \) has the value

\[
aK \text{ or } -K \frac{dv}{dx}.
\]

139. The propositions which we have proved in the preceding articles apply also to the case in which the instantaneous action of a molecule is exerted in the interior of the mass up to an appreciable distance. In this case, we must suppose that the cause which maintains the external layers of the body in the state expressed by the linear equation, affects the mass up to a finite depth. All observation concurs to prove that in solids and liquids the distance in question is extremely small.

140. Theorem III. If the temperatures at the points of a solid are expressed by the equation \( v = f(x, y, z, t) \), in which \( x, y, z \) are the co-ordinates of a molecule whose temperature is equal to \( v \) after the lapse of a time \( t \); the flow of heat which crosses part of a plane traced in the solid, perpendicular to one of the three axes, is no longer constant; its value is different for different parts of the plane, and it varies also with the time. This variable quantity may be determined by analysis.
Let \( \omega \) be an infinitely small circle whose centre coincides with the point \( m \) of the solid, and whose plane is perpendicular to the vertical co-ordinate \( z \); during the instant \( dt \) there will flow across this circle a certain quantity of heat which will pass from the part of the circle below the plane of the circle into the upper part. This flow is composed of all the rays of heat which depart from a lower point and arrive at an upper point, by crossing a point of the small surface \( \omega \). We proceed to shew that the expression of the value of the flow is \( \frac{dv}{dz} \omega dt \).

Let us denote by \( x', y', z' \) the coordinates of the point \( m \) whose temperature is \( v' \); and suppose all the other molecules to be referred to this point \( m \) chosen as the origin of new axes parallel to the former axes: let \( \xi, \eta, \zeta \), be the three co-ordinates of a point referred to the origin \( m \); in order to express the actual temperature \( w \) of a molecule infinitely near to \( m \), we shall have the linear equation

\[
w = v' + \xi \frac{dv'}{dx} + \eta \frac{dv'}{dy} + \zeta \frac{dv'}{dz}.
\]

The coefficients \( v', \frac{dv'}{dx}, \frac{dv'}{dy}, \frac{dv'}{dz} \) are the values which are found by substituting in the functions \( v, \frac{dv}{dx}, \frac{dv}{dy}, \frac{dv}{dz} \), for the variables \( x, y, z \), the constant quantities \( x', y', z' \), which measure the distances of the point \( m \) from the first three axes of \( x, y, \) and \( z \).

Suppose now that the point \( m \) is also an internal molecule of a rectangular prism, enclosed between six planes perpendicular to the three axes whose origin is \( m \); that \( w \) the actual temperature of each molecule of this prism, whose dimensions are finite, is expressed by the linear equation \( w = A + a\xi + b\eta + c\zeta \), and that the six faces which bound the prism are maintained at the fixed temperatures which the last equation assigns to them. The state of the internal molecules will also be permanent, and a quantity of heat measured by the expression \(-Kcw\omega dt\) will flow during the instant \( dt \) across the circle \( \omega \).

This arranged, if we take as the values of the constants \( A, a, b, c, \) the quantities \( v', \frac{dv'}{dx}, \frac{dv'}{dy}, \frac{dv'}{dz} \), the fixed state of the
Thus the molecules infinitely near to the point \( m \) will have, during the instant \( dt \), the same actual temperature in the solid whose state is variable, and in the prism whose state is constant. Hence the flow which exists at the point \( m \), during the instant \( dt \), across the infinitely small circle \( \omega \), is the same in either solid; it is therefore expressed by \(- K \frac{dv'}{dz} \omega dt\).

From this we derive the following proposition

*If in a solid whose internal temperatures vary with the time, by virtue of the action of the molecules, we trace any straight line whatever, and erect (see fig. 5), at the different points of this line, the ordinates \( pm \) of a plane curve equal to the temperatures of these points taken at the same moment; the flow of heat, at each point \( p \) of the straight line, will be proportional to the tangent of the angle \( \alpha \) which the element of the curve makes with the parallel to the abscissa \( o \); that is to say, if at the point \( p \) we place the centre of an infinitely small circle \( \omega \) perpendicular to the line, the quantity of heat which has flowed during the instant \( dt \), across this circle, in the direction in which the abscissa \( op \) increase, will be measured by the product of four factors, which are, the tangent of the angle \( \alpha \), a constant coefficient \( K \), the area \( \omega \) of the circle, and the duration \( dt \) of the instant.*

141. **Corollary.** If we represent by \( e \) the abscissa of this curve or the distance of a point \( p \) of the straight line from a
fixed point o, and by v the ordinate which represents the temperature of the point p, v will vary with the distance e and will be a certain function f(e) of that distance; the quantity of heat which would flow across the circle o, placed at the point p perpendicular to the line, will be \(-K \frac{dv}{de} \omega dt\), or

\[-Kf'(e) \omega dt,\]

denoting the function \(\frac{df(e)}{de}\) by \(f'(e)\).

We may express this result in the following manner, which facilitates its application.

To obtain the actual flow of heat at a point p of a straight line drawn in a solid, whose temperatures vary by action of the molecules, we must divide the difference of the temperatures at two points infinitely near to the point p by the distance between these points. The flow is proportional to the quotient.

142. Theorem IV. From the preceding Theorems it is easy to deduce the general equations of the propagation of heat.

Suppose the different points of a homogeneous solid of any form whatever, to have received initial temperatures which vary successively by the effect of the mutual action of the molecules, and suppose the equation \(v = f(x, y, z, t)\) to represent the successive states of the solid, it may now be shewn that v a function of four variables necessarily satisfies the equation

\[
\frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} \right).
\]

In fact, let us consider the movement of heat in a molecule enclosed between six planes at right angles to the axes of x, y, and z; the first three of these planes pass through the point m whose coordinates are x, y, z, the other three pass through the point m', whose coordinates are \(x + dx, y + dy, z + dz\).

During the instant \(dt\), the molecule receives, across the lower rectangle \(dxdy\), which passes through the point m, a quantity of heat equal to \(-K dxdy \frac{dv}{dz} dt\). To obtain the quantity which escapes from the molecule by the opposite face, it is sufficient to change z into \(z + dz\) in the preceding expression,
that is to say, to add to this expression its own differential taken with respect to $z$ only; we then have

$$-K dx \frac{dv}{dz} \frac{dv}{dt} - K dx dy \frac{d}{dz} \frac{dv}{dz} dx dt$$

as the value of the quantity which escapes across the upper rectangle. The same molecule receives also across the first rectangle $dz \, dx$ which passes through the point $m$, a quantity of heat equal to $-K \frac{dv}{dy} dz \, dx \, dt$; and if we add to this expression its own differential taken with respect to $y$ only, we find that the quantity which escapes across the opposite face $dz \, dx$ is expressed by

$$- K \frac{dv}{dy} dz \, dx \, dt - K \frac{dv}{dy} dy \, dz \, dx \, dt.$$

Lastly, the molecule receives through the first rectangle $dy \, dz$ a quantity of heat equal to $-K \frac{dv}{dx} dy \, dz \, dt$, and that which it loses across the opposite rectangle which passes through $m'$ is expressed by

$$-K \frac{dv}{dx} dy \, dz \, dt - K \frac{dv}{dx} dx \, dy \, dz \, dt.$$

We must now take the sum of the quantities of heat which the molecule receives and subtract from it the sum of those which it loses. Hence it appears that during the instant $dt$, a total quantity of heat equal to

$$K \left( \frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} \right) dx \, dy \, dz \, dt$$

accumulates in the interior of the molecule. It remains only to obtain the increase of temperature which must result from this addition of heat.

$D$ being the density of the solid, or the weight of unit of volume, and $C$ the specific capacity, or the quantity of heat which raises the unit of weight from the temperature 0 to the temperature 1; the product $CD \, dx \, dy \, dz$ expresses the quantity
of heat required to raise from 0 to 1 the molecule whose volume is \(dx\,dy\,dz\). Hence dividing by this product the quantity of heat which the molecule has just acquired, we shall have its increase of temperature. Thus we obtain the general equation

\[
\frac{dv}{dt} = K \frac{1}{CD} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \]

which is the equation of the propagation of heat in the interior of all solid bodies.

143. Independently of this equation the system of temperatures is often subject to several definite conditions, of which no general expression can be given, since they depend on the nature of the problem.

If the dimensions of the mass in which heat is propagated are finite, and if the surface is maintained by some special cause in a given state; for example, if all its points retain, by virtue of that cause, the constant temperature 0, we shall have, denoting the unknown function \(v\) by \(\phi(x, y, z, t)\), the equation of condition \(\phi(x, y, z, t) = 0\); which must be satisfied by all values of \(x, y, z\) which belong to points of the external surface, whatever be the value of \(t\). Further, if we suppose the initial temperatures of the body to be expressed by the known function \(F(x, y, z)\), we have also the equation \(\phi(x, y, z, 0) = F(x, y, z)\); the condition expressed by this equation must be fulfilled by all values of the co-ordinates \(x, y, z\) which belong to any point whatever of the solid.

144. Instead of submitting the surface of the body to a constant temperature, we may suppose the temperature not to be the same at different points of the surface, and that it varies with the time according to a given law; which is what takes place in the problem of terrestrial temperature. In this case the equation relative to the surface contains the variable \(t\).

145. In order to examine by itself, and from a very general point of view, the problem of the propagation of heat, the solid whose initial state is given must be supposed to have all its dimensions infinite; no special condition disturbs then the dif-
fusion of heat, and the law to which this principle is submitted becomes more manifest; it is expressed by the general equation

$$\frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{dv}{dz} \right),$$

to which must be added that which relates to the initial arbitrary state of the solid.

Suppose the initial temperature of a molecule, whose coordinates are \(x, y, z\), to be a known function \(F(x, y, z)\), and denote the unknown value \(v\) by \(\phi(x, y, z, t)\), we shall have the definite equation \(\phi(x, y, z, 0) = F(x, y, z)\); thus the problem is reduced to the integration of the general equation (A) in such a manner that it may agree, when the time is zero, with the equation which contains the arbitrary function \(F\).

SECTION VII.

General equation relative to the surface.

146. If the solid has a definite form, and if its original heat is dispersed gradually into atmospheric air maintained at a constant temperature, a third condition relative to the state of the surface must be added to the general equation (A) and to that which represents the initial state.

We proceed to examine, in the following articles, the nature of the equation which expresses this third condition.

Consider the variable state of a solid whose heat is dispersed into air, maintained at the fixed temperature 0. Let \(\omega\) be an infinitely small part of the external surface, and \(\mu\) a point of \(\omega\), through which a normal to the surface is drawn; different points of this line have at the same instant different temperatures.

Let \(v\) be the actual temperature of the point \(\mu\), taken at a definite instant, and \(w\) the corresponding temperature of a point \(v\) of the solid taken on the normal, and distant from \(\mu\) by an infinitely small quantity \(\alpha\). Denote by \(x, y, z\) the co-ordinates of the point \(\mu\), and those of the point \(v\) by \(x + \delta x, y + \delta y, z + \delta z\); let \(f(x, y, z) = 0\) be the known equation to the surface of the solid, and \(v = \phi(x, y, z, t)\) the general equation which ought to give the
value of \(v\) as a function of the four variables \(x, y, z, t\). Differentiating the equation \(f(x, y, z) = 0\), we shall have
\[m\, dx + n\, dy + p\, dz = 0;\]
\(m, n, p\) being functions of \(x, y, z\).

It follows from the corollary enunciated in Article 141, that the flow in direction of the normal, or the quantity of heat which during the instant \(dt\) would cross the surface \(\omega\), if it were placed at any point whatever of this line, at right angles to its direction, is proportional to the quotient which is obtained by dividing the difference of temperature of two points infinitely near by their distance. Hence the expression for the flow at the end of the normal is

\[-K\, \frac{w - v}{\alpha}\, \omega dt;\]

\(K\) denoting the specific conducibility of the mass. On the other hand, the surface \(\omega\) permits a quantity of heat to escape into the air, during the time \(dt\), equal to \(h\, v\, \omega dt\); \(h\) being the conducibility relative to atmospheric air. Thus the flow of heat at the end of the normal has two different expressions, that is to say:

\[h\, v\, \omega dt\] and \[-K\, \frac{w - v}{\alpha}\, \omega dt;\]

hence these two quantities are equal; and it is by the expression of this equality that the condition relative to the surface is introduced into the analysis.

147. We have
\[w = v + \delta v = v + \frac{dv}{dx}\, \delta x + \frac{dv}{dy}\, \delta y + \frac{dv}{dz}\, \delta z.\]

Now, it follows from the principles of geometry, that the coordinates \(\delta x, \delta y, \delta z\), which fix the position of the point \(v\) of the normal relative to the point \(\mu\), satisfy the following conditions:
\[p\, \delta x = m\, \delta z, \quad p\, \delta y = n\, \delta z.\]

We have therefore
\[w - v = \frac{1}{p} \left(m\, \frac{dv}{dx} + n\, \frac{dv}{dy} + p\, \frac{dv}{dz}\right) \delta z:\]
we have also
\[ \alpha = \sqrt{\delta x^2 + \delta y^2 + \delta z^2} = \frac{1}{p} (m^2 + n^2 + p^2)^{\frac{1}{2}} \delta z, \]
or \[ \alpha = \frac{q}{p} \delta z, \] denoting by \( q \) the quantity \((m^2 + n^2 + p^2)^{\frac{1}{2}}\),

hence
\[ \frac{w - v}{\alpha} = \left( m \frac{dv}{dx} + n \frac{dv}{dy} + p \frac{dv}{dz} \right) \frac{1}{q}; \]

consequently the equation
\[ h v \omega dt = - k \left( \frac{w - v}{\alpha} \right) \omega dt \]
becomes the following\(^1\):
\[ m \frac{dv}{dx} + n \frac{dv}{dy} + p \frac{dv}{dz} + \frac{h}{K} v q = 0 \] ....... (B).

This equation is definite and applies only to points at the surface; it is that which must be added to the general equation of the propagation of heat (A), and to the condition which determines the initial state of the solid; \( m, n, p, q \), are known functions of the co-ordinates of the points on the surface.

148. The equation (B) signifies in general that the decrease of the temperature, in the direction of the normal, at the boundary of the solid, is such that the quantity of heat which tends to escape by virtue of the action of the molecules, is equivalent always to that which the body must lose in the medium.

The mass of the solid might be imagined to be prolonged, in such a manner that the surface, instead of being exposed to the air, belonged at the same time to the body which it bounds, and to the mass of a solid envelope which contained it. If, on this hypothesis, any cause whatever regulated at every instant the decrease of the temperatures in the solid envelope, and determined it in such a manner that the condition expressed by the equation (B) was always satisfied, the action of the envelope would take the

\(^1\) Let \( N \) be the normal,
\[ K \frac{dv}{dN} + hv = 0, \]
\[ \frac{dv}{dN} = \frac{m dv}{q dx} + \text{&c.}; \]

the rest as in the text. [R. L. E.]
place of that of the air, and the movement of heat would be the
same in either case: we can suppose then that this cause exists,
and determine on this hypothesis the variable state of the solid;
which is what is done in the employment of the two equations
(A) and (B).

By this it is seen how the interruption of the mass and the
action of the medium, disturb the diffusion of heat by submitting
it to an accidental condition.

149. We may also consider the equation (B), which relates
to the state of the surface under another point of view: but we
must first derive a remarkable consequence from Theorem III.
(Art. 140). We retain the construction referred to in the corollary
of the same theorem (Art. 141). Let \( x, y, z \) be the co-ordinates
of the point \( p \), and

\[
x + \delta x, \ y + \delta y, \ z + \delta z
\]

those of a point \( q \) infinitely near to \( p \), and taken on the straight
line in question: if we denote by \( v \) and \( w \) the temperatures of the
two points \( p \) and \( q \) taken at the same instant, we have

\[
w = v + \delta v = v + \frac{dv}{dx} \delta x + \frac{dv}{dy} \delta y + \frac{dv}{dz} \delta z;
\]

hence the quotient

\[
\frac{\delta v}{\delta e} = \frac{dv}{dx} \frac{\delta x}{\delta e} + \frac{dv}{dy} \frac{\delta y}{\delta e} + \frac{dv}{dz} \frac{\delta z}{\delta e}, \quad \text{and} \quad \delta e = \sqrt{\delta x^2 + \delta y^2 + \delta z^2};
\]

thus the quantity of heat which flows across the surface \( \omega \) placed
at the point \( m \), perpendicular to the straight line, is

\[
-K \omega dt \left\{ \frac{dv}{dx} \frac{\delta x}{\delta e} + \frac{dv}{dy} \frac{\delta y}{\delta e} + \frac{dv}{dz} \frac{\delta z}{\delta e} \right\}.
\]

The first term is the product of \(-K \frac{dv}{dx}\) by \( dt \) and by \( \omega \frac{\delta x}{\delta e} \). The latter quantity is, according to the principles of geometry, the
area of the projection of \( \omega \) on the plane of \( y \) and \( z \); thus the
product represents the quantity of heat which would flow across
the area of the projection, if it were placed at the point \( p \) perpen-
dicular to the axis of \( x \).
The second term \(- K \frac{dv}{dy} \omega \frac{\delta y}{\delta \epsilon} dt\) represents the quantity of heat which would cross the projection of \(\omega\), made on the plane of \(x\) and \(z\), if this projection were placed parallel to itself at the point \(p\).

Lastly, the third term \(- K \frac{dv}{dz} \omega \frac{\delta z}{\delta \epsilon} dt\) represents the quantity of heat which would flow during the instant \(dt\), across the projection of \(\omega\) on the plane of \(x\) and \(y\), if this projection were placed at the point \(p\), perpendicular to the co-ordinate \(z\).

By this it is seen that the quantity of heat which flows across every infinitely small part of a surface drawn in the interior of the solid, can always be decomposed into three other quantities of flow, which penetrate the three orthogonal projections of the surface, along the directions perpendicular to the planes of the projections. The result gives rise to properties analogous to those which have been noticed in the theory of forces.

150. The quantity of heat which flows across a plane surface \(\omega\), infinitely small, given in form and position, being equivalent to that which would cross its three orthogonal projections, it follows that, if in the interior of the solid an element be imagined of any form whatever, the quantities of heat which pass into this polyhedron by its different faces, compensate each other reciprocally: or more exactly, the sum of the terms of the first order, which enter into the expression of the quantities of heat received by the molecule, is zero; so that the heat which is in fact accumulated in it, and makes its temperature vary, cannot be expressed except by terms infinitely smaller than those of the first order.

This result is distinctly seen when the general equation (A) has been established, by considering the movement of heat in a prismatic molecule (Articles 127 and 142); the demonstration may be extended to a molecule of any form whatever, by substituting for the heat received through each face, that which its three projections would receive.

In other respects it is necessary that this should be so: for, if one of the molecules of the solid acquired during each instant a quantity of heat expressed by a term of the first order, the variation of its temperature would be infinitely greater than that of
other molecules, that is to say, during each infinitely small instant its temperature would increase or decrease by a finite quantity, which is contrary to experience.

151. We proceed to apply this remark to a molecule situated at the external surface of the solid.

Through a point $a$ (see fig. 6), taken on the plane of $x$ and $y$, draw two planes perpendicular, one to the axis of $x$ the other to the axis of $y$. Through a point $b$ of the same plane, infinitely near to $a$, draw two other planes parallel to the two preceding planes; the ordinates $z$, raised at the points $a, b, c, d$, up to the external surface of the solid, will mark on this surface four points $a', b', c', d'$, and will be the edges of a truncated prism, whose base is the rectangle $abcd$. If through the point $a'$ which denotes the least elevated of the four points $a', b', c', d'$, a plane be drawn parallel to that of $x$ and $y$, it will cut off from the truncated prism a molecule, one of whose faces, that is to say $a'b'c'd'$, coincides with the surface of the solid. The values of the four ordinates $aa', cc', dd', bb'$ are the following:

$$aa' = z,$$

$$cc' = z + \frac{dz}{dx} dx,$$

$$dd' = z + \frac{dz}{dy} dy,$$

$$bb' = z + \frac{dz}{dx} dx + \frac{dz}{dy} dy.$$
152. One of the faces perpendicular to $x$ is a triangle, and the opposite face is a trapezium. The area of the triangle is

$$\frac{1}{2} \, dy \frac{dz}{dy} \, dy,$$

and the flow of heat in the direction perpendicular to this surface being $-K \frac{dv}{dx}$ we have, omitting the factor $dt$,

$$-K \frac{dv}{dx} \frac{1}{2} \, dy \frac{dz}{dy} \, dy,$$

as the expression of the quantity of heat which in one instant passes into the molecule, across the triangle in question.

The area of the opposite face is

$$\frac{1}{2} \, dy \left( \frac{dz}{dx} \, dx + \frac{dz}{dx} \, dx + \frac{dz}{dy} \, dy \right),$$

and the flow perpendicular to this face is also $-K \frac{dv}{dx}$, suppressing terms of the second order infinitely smaller than those of the first; subtracting the quantity of heat which escapes by the second face from that which enters by the first we find

$$K \frac{dv}{dx} \frac{dz}{dy} \, dx \, dy.$$

This term expresses the quantity of heat the molecule receives through the faces perpendicular to $x$.

It will be found, by a similar process, that the same molecule receives, through the faces perpendicular to $y$, a quantity of heat equal to $K \frac{dv}{dy} \frac{dz}{dy} \, dx \, dy$.

The quantity of heat which the molecule receives through the rectangular base is $-K \frac{dv}{dz} \, dx \, dy$. Lastly, across the upper surface $a'b'c'd'$, a certain quantity of heat is permitted to escape, equal to the product of $hv$ into the extent $\omega$ of that surface. The value of $\omega$ is, according to known principles, the same as that of $dz \, dy$ multiplied by the ratio $\frac{\epsilon}{z}$; $\epsilon$ denoting the length of the normal between the external surface and the plane of $x$ and $y$, and

$$\epsilon = z \left\{ 1 + \left( \frac{dz}{dx} \right)^2 + \left( \frac{dz}{dy} \right)^2 \right\}^{\frac{1}{2}},$$
hence the molecule loses across its surface \(a'b'c'd'\) a quantity of heat equal to \(hv \, dz \, dy \frac{\epsilon}{z}\).

Now, the terms of the first order which enter into the expression of the total quantity of heat acquired by the molecule, must cancel each other, in order that the variation of temperature may not be at each instant a finite quantity; we must then have the equation

\[
K \left( \frac{dv}{dx} \frac{dz}{dx} \, dx \, dy + \frac{dv}{dy} \frac{dz}{dy} \, dx \, dy - \frac{dv}{dz} \frac{dx}{dy} \, dy \right) - hv \frac{\epsilon}{z} \, dz \, dy = 0,
\]

or

\[
\frac{h}{K} v \frac{\epsilon}{z} = \frac{dv}{dx} \frac{dz}{dx} + \frac{dv}{dy} \frac{dz}{dy} - \frac{dv}{dz}.
\]

153. Substituting for \(\frac{dz}{dx}\) and \(\frac{dz}{dy}\) their values derived from the equation

\[
mdx + ndy + pdz = 0,
\]

and denoting by \(q\) the quantity

\[
(m^2 + n^2 + p^2),
\]

we have

\[
K \left( m \frac{dv}{dx} + n \frac{dv}{dy} + p \frac{dv}{dz} \right) + hvq = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldot

154. To arrive at equation (B), we in fact consider one of the molecules whose base is in the surface of the solid, as a vessel which receives or loses heat through its different faces. The equation signifies that all the terms of the first order which
enter into the expression of the heat acquired cancel each other; so that the gain of heat cannot be expressed except by terms of the second order. We may give to the molecule the form, either of a right prism whose axis is normal to the surface of the solid, or that of a truncated prism, or any form whatever.

The general equation (A), (Art. 142) supposes that all the terms of the first order cancel each other in the interior of the mass, which is evident for prismatic molecules enclosed in the solid. The equation (B), (Art. 147) expresses the same result for molecules situated at the boundaries of bodies.

Such are the general points of view from which we may look at this part of the theory of heat.

The equation \( \frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} \right) \) represents the movement of heat in the interior of bodies. It enables us to ascertain the distribution from instant to instant in all substances solid or liquid; from it we may derive the equation which belongs to each particular case.

In the two following articles we shall make this application to the problem of the cylinder, and to that of the sphere.

**SECTION VIII.**

*Application of the general equations.*

155. Let us denote the variable radius of any cylindrical envelope by \( r \), and suppose, as formerly, in Article 118, that all the molecules equally distant from the axis have at each instant a common temperature; \( v \) will be a function of \( r \) and \( t \); \( r \) is a function of \( y, z \), given by the equation \( r^2 = y^2 + z^2 \). It is evident in the first place that the variation of \( v \) with respect to \( x \) is nul; thus the term \( \frac{d^2v}{dx^2} \) must be omitted. We shall have then, according to the principles of the differential calculus, the equations

\[
\frac{dv}{dy} = \frac{dv}{dr} \frac{dr}{dy} \quad \text{and} \quad \frac{d^2v}{dy^2} = \frac{d^2v}{dr^2} \left( \frac{dr}{dy} \right)^2 + \frac{dv}{dr} \frac{d^2r}{dy^2},
\]

\[
\frac{dv}{dz} = \frac{dv}{dr} \frac{dr}{dz} \quad \text{and} \quad \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} \left( \frac{dr}{dz} \right)^2 + \frac{dv}{dr} \frac{d^2r}{dz^2};
\]
whence
\[ \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} \left\{ \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 + \frac{dv}{dr} \left( \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} \right) \right\} \dots (a). \]

In the second member of the equation, the quantities
\[ \frac{dr}{dy}, \frac{dr}{dz}, \frac{d^2r}{dy^2}, \frac{d^2r}{dz^2}, \]
must be replaced by their respective values; for which purpose we derive from the equation \( y^2 + z^2 = r^2 \),
\[ y = r \frac{dr}{dy} \quad \text{and} \quad 1 = \left( \frac{dr}{dy} \right)^2 + r \frac{d^2r}{dy^2}, \]
\[ z = r \frac{dr}{dz} \quad \text{and} \quad 1 = \left( \frac{dr}{dz} \right)^2 + r \frac{d^2r}{dz^2}, \]
and consequently
\[ y^2 + z^2 = r^2 \left\{ \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 \right\}, \]
\[ 2 = \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 + r \left[ \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} \right]. \]
The first equation, whose first member is equal to \( r^2 \), gives
\[ \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 = 1 \dots (b) ; \]
the second gives, when we substitute for
\[ \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 \]
its value 1,
\[ \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} = \frac{1}{r} \dots (c). \]

If the values given by equations \((b)\) and \((c)\) be now substituted in \((a)\), we have
\[ \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \cdot \]

Hence the equation which expresses the movement of heat in the cylinder, is
\[ \frac{dv}{dt} = \frac{K}{CD} \left( \frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \right), \]
as was found formerly, Art. 119.
We might also suppose that particles equally distant from the centre have not received a common initial temperature; in this case we should arrive at a much more general equation.

156. To determine, by means of equation (A), the movement of heat in a sphere which has been immersed in a liquid, we shall regard $v$ as a function of $r$ and $t$; $r$ is a function of $x, y, z$, given by the equation

$$r^2 = x^2 + y^2 + z^2,$$

$r$ being the variable radius of an envelope. We have then

$$\frac{dv}{dx} = \frac{dv}{dr} \frac{dr}{dx} \quad \text{and} \quad \frac{d^2v}{dx^2} = \frac{d^2v}{dr^2} \left(\frac{dr}{dx}\right)^2 + \frac{dv}{dr} \frac{d^2r}{dx^2},$$

$$\frac{dv}{dy} = \frac{dv}{dr} \frac{dr}{dy} \quad \text{and} \quad \frac{d^2v}{dy^2} = \frac{d^2v}{dr^2} \left(\frac{dr}{dy}\right)^2 + \frac{dv}{dr} \frac{d^2r}{dy^2},$$

$$\frac{dv}{dz} = \frac{dv}{dr} \frac{dr}{dz} \quad \text{and} \quad \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} \left(\frac{dr}{dz}\right)^2 + \frac{dv}{dr} \frac{d^2r}{dz^2}.$$ 

Making these substitutions in the equation

$$\frac{dv}{dt} = \frac{K}{\mathcal{C} \mathcal{D}} \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

we shall have

$$\frac{dv}{dt} = \frac{K}{\mathcal{C} \mathcal{D}} \left[\frac{d^2v}{dr^2} \left(\frac{dr}{dx}\right)^2 + \left(\frac{dr}{dy}\right)^2 + \left(\frac{dr}{dz}\right)^2\right] + \frac{dv}{dr} \left(\frac{d^2r}{dx^2} + \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2}\right) (a).$$

The equation $x^2 + y^2 + z^2 = r^2$ gives the following results;

$$x = r \frac{dr}{dx} \quad \text{and} \quad 1 = \left(\frac{dr}{dx}\right)^2 + r \frac{d^2r}{dx^2},$$

$$y = r \frac{dr}{dy} \quad \text{and} \quad 1 = \left(\frac{dr}{dy}\right)^2 + r \frac{d^2r}{dy^2},$$

$$z = r \frac{dr}{dz} \quad \text{and} \quad 1 = \left(\frac{dr}{dz}\right)^2 + r \frac{d^2r}{dz^2}.$$ 

The three equations of the first order give:

$$x^2 + y^2 + z^2 = r^2 \left(\frac{dr}{dx}\right)^2 + \left(\frac{dr}{dy}\right)^2 + \left(\frac{dr}{dz}\right)^2.$$
The three equations of the second order give:

\[ 3 = \left( \frac{dr}{dx} \right)^2 + \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 + r \left( \frac{d^2r}{dx^2} + \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} \right) \]

and substituting for

\[ \left( \frac{dr}{dx} \right)^2 + \left( \frac{dr}{dy} \right)^2 + \left( \frac{dr}{dz} \right)^2 \]

its value 1, we have

\[ \frac{d^2r}{dx^2} + \frac{d^2r}{dy^2} + \frac{d^2r}{dz^2} = \frac{2}{r} \cdot \frac{dr}{d^r} \cdot \frac{dr}{d^r} \cdot \frac{dr}{d^r} \cdot \frac{dr}{d^r} \cdot \frac{dr}{d^r} \]

Making these substitutions in the equation (a) we have the equation

\[ \frac{dv}{dt} = K \left( \frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} \right), \]

which is the same as that of Art. 114.

The equation would contain a greater number of terms, if we supposed molecules equally distant from the centre not to have received the same initial temperature.

We might also deduce from the definite equation (B), the equations which express the state of the surface in particular cases, in which we suppose solids of given form to communicate their heat to the atmospheric air; but in most cases these equations present themselves at once, and their form is very simple, when the co-ordinates are suitably chosen.

SECTION IX.

General Remarks.

157. The investigation of the laws of movement of heat in solids now consists in the integration of the equations which we have constructed; this is the object of the following chapters. We conclude this chapter with general remarks on the nature of the quantities which enter into our analysis.

In order to measure these quantities and express them numerically, they must be compared with different kinds of units, five
in number, namely, the unit of length, the unit of time, that of temperature, that of weight, and finally the unit which serves to measure quantities of heat. For the last unit, we might have chosen the quantity of heat which raises a given volume of a certain substance from the temperature 0 to the temperature 1. The choice of this unit would have been preferable in many respects to that of the quantity of heat required to convert a mass of ice of a given weight, into an equal mass of water at 0, without raising its temperature. We have adopted the last unit only because it had been in a manner fixed beforehand in several works on physics; besides, this supposition would introduce no change into the results of analysis.

158. The specific elements which in every body determine the measurable effects of heat are three in number, namely, the conducibility proper to the body, the conducibility relative to the atmospheric air, and the capacity for heat. The numbers which express these quantities are, like the specific gravity, so many natural characters proper to different substances.

We have already remarked, Art. 36, that the conducibility of the surface would be measured in a more exact manner, if we had sufficient observations on the effects of radiant heat in spaces deprived of air.

It may be seen, as has been mentioned in the first section of Chapter I, Art. 11, that only three specific coefficients, $K$, $h$, $C$, enter into the investigation; they must be determined by observation; and we shall point out in the sequel the experiments adapted to make them known with precision.

159. The number $C$ which enters into the analysis, is always multiplied by the density $D$, that is to say, by the number of units of weight which are equivalent to the weight of unit of volume; thus the product $CD$ may be replaced by the coefficient $c$. In this case we must understand by the specific capacity for heat, the quantity required to raise from temperature 0 to temperature 1 unit of volume of a given substance, and not unit of weight of that substance.

With the view of not departing from the common definition, we have referred the capacity for heat to the weight and not to
the volume; but it would be preferable to employ the coefficient $c$
which we have just defined; magnitudes measured by the unit
of weight would not then enter into the analytical expressions:
we should have to consider only, 1st, the linear dimension $x$, the
temperature $v$, and the time $t$; 2nd, the coefficients $c$, $h$, and $K$.
The three first quantities are undetermined, and the three others
are, for each substance, constant elements which experiment
determines. As to the unit of surface and the unit of volume,
they are not absolute, but depend on the unit of length.

160. It must now be remarked that every undetermined
magnitude or constant has one dimension proper to itself, and
that the terms of one and the same equation could not be com-
pared, if they had not the same exponent of dimension. We have
introduced this consideration into the theory of heat, in order to
make our definitions more exact, and to serve to verify the
analysis; it is derived from primary notions on quantities; for
which reason, in geometry and mechanics, it is the equivalent
of the fundamental lemmas which the Greeks have left us with-
out proof.

161. In the analytical theory of heat, every equation $(E)$
expresses a necessary relation between the existing magnitudes
$x, t, v, c, h, K$. This relation depends in no respect on the choice
of the unit of length, which from its very nature is contingent,
that is to say, if we took a different unit to measure the linear
dimensions, the equation $(E)$ would still be the same. Suppose
then the unit of length to be changed, and its second value to be
equal to the first divided by $m$. Any quantity whatever $x$ which
in the equation $(E)$ represents a certain line $ab$, and which, con-
sequently, denotes a certain number of times the unit of length,
becomes $mx$, corresponding to the same length $ab$; the value $t$
of the time, and the value $v$ of the temperature will not be
changed; the same is not the case with the specific elements
$h, K, c$: the first, $h$, becomes $\frac{h}{m}$; for it expresses the quantity of
heat which escapes, during the unit of time, from the unit of sur-
face at the temperature 1. If we examine attentively the nature
of the coefficient $K$, as we have defined it in Articles 68 and 135,
we perceive that it becomes \( \frac{K}{m} \); for the flow of heat varies directly as the area of the surface, and inversely as the distance between two infinite planes (Art. 72). As to the coefficient \( c \) which represents the product \( CD \), it also depends on the unit of length and becomes \( \frac{c}{m^3} \); hence equation \((E)\) must undergo no change when we write \( mx \) instead of \( x \), and at the same time \( \frac{K}{m} \), \( \frac{h}{m^2} \), \( \frac{c}{m^3} \), instead of \( K \), \( h \), \( c \); the number \( m \) disappears after these substitutions: thus the dimension of \( x \) with respect to the unit of length is 1, that of \( K \) is \(-1\), that of \( h \) is \(-2\), and that of \( c \) is \(-3\). If we attribute to each quantity its own exponent of dimension, the equation will be homogeneous, since every term will have the same total exponent. Numbers such as \( S \), which represent surfaces or solids, are of two dimensions in the first case, and of three dimensions in the second. Angles, sines, and other trigonometrical functions, logarithms or exponents of powers, are, according to the principles of analysis, absolute numbers which do not change with the unit of length; their dimensions must therefore be taken equal to 0, which is the dimension of all abstract numbers.

If the unit of time, which was at first 1, becomes \( \frac{1}{n} \), the number \( t \) will become \( nt \), and the numbers \( x \) and \( v \) will not change. The coefficients \( K \), \( h \), \( c \) will become \( K \), \( \frac{h}{n} \), \( \frac{c}{n} \). Thus the dimensions of \( x \), \( t \), \( v \) with respect to the unit of time are 0, 1, 0, and those of \( K \), \( h \), \( c \) are \(-1\), \(-1\), \(0\).

If the unit of temperature be changed, so that the temperature 1 becomes that which corresponds to an effect other than the boiling of water; and if that effect requires a less temperature, which is to that of boiling water in the ratio of \( 1 \) to the number \( p \); \( v \) will become \( vp \), \( x \) and \( t \) will keep their values, and the coefficients \( K \), \( h \), \( c \) will become \( \frac{K}{p} \), \( \frac{h}{p} \), \( \frac{c}{p} \).

The following table indicates the dimensions of the three undetermined quantities and the three constants, with respect to each kind of unit.

F. H. 9
162. If we retained the coefficients $C$ and $D$, whose product has been represented by $c$, we should have to consider the unit of weight, and we should find that the exponent of dimension, with respect to the unit of length, is $-3$ for the density $D$, and 0 for $C$.

On applying the preceding rule to the different equations and their transformations, it will be found that they are homogeneous with respect to each kind of unit, and that the dimension of every angular or exponential quantity is nothing. If this were not the case, some error must have been committed in the analysis, or abridged expressions must have been introduced.

If, for example, we take equation (b) of Art. 105,

$$\frac{dv}{dt} = \frac{K}{CD} \frac{dv}{dx} - \frac{hl}{CDS} v,$$

we find that, with respect to the unit of length, the dimension of each of the three terms is 0; it is 1 for the unit of temperature, and $-1$ for the unit of time.

In the equation $v = Ae^{-x\sqrt{\frac{2h}{Kl}}}$ of Art. 76, the linear dimension of each term is 0, and it is evident that the dimension of the exponent $x\sqrt{\frac{2h}{Kl}}$ is always nothing, whatever be the units of length, time, or temperature.